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THREE-DIMENSIONAL INTERACTIONS OF A HALF-PLANE CRACK WITH POINT FORCES, DIPOLES AND MOMENTS

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Abstract—Closed form solutions in terms of elementary functions are derived for stress intensity factors along the edge of a half-plane crack induced by point forces, dipoles, centers of expansion and moments. Locations and orientations of these "stress sources" are arbitrary. The solid is assumed to be either transversely isotropic (with the crack parallel to the plane of isotropy) or isotropic. © 1997 Published by Elsevier Science Ltd.

1. INTRODUCTION

Stress intensity factors (SIFs) induced on a half-plane crack by point forces, dipoles, moments, centers of dilatation and rotation are derived in elementary functions. Positions and orientations of these stress sources with respect to the crack are arbitrary. The material is assumed to be either transversely isotropic (with the crack plane being parallel to the plane of isotropy) or isotropic.

Aside from being of importance of their own, such solutions are of interest for applications, since the mentioned stress sources may, under certain conditions, model defects like microcracks, cavities, foreign particles, dislocations, etc. The importance of these stress sources is also due to the fact that, they represent an arbitrary system of forces distributed in a small volume V: at distances from V much larger than the size of V, the displacement and stress can be represented, to within small values of higher order, as a sum of the fields generated by the *resultants* (principal vector, resultant moment and three mutually orthogonal dipoles), see discussion of Karapetian and Kachanov (1996).

The present article continues the work of Karapetian and Hanson (1994) and Karapetian and Kachanov (1996) on a circular crack. We utilize recent results of Fabrikant *et al.* (1993, 1995) where the elastic fields generated by a half-plane crack loaded by pair of equal and opposite point forces applied at the crack faces were derived in closed form. These results, coupled with the reciprocity theorem and with the representations of SIFs in terms of the displacement discontinuities on crack faces (Fabrikant, 1989), allow one to derive expressions in elementary functions for the crack opening displacements (COD) and SIFs for a half-plane crack due to a point force, arbitrarily located and oriented. Using these expressions, we solve the problem of interaction of a half-plane crack with (arbitrarily located and oriented) stress sources like dipoles, centers of dilatation, moments and centers of rotation.

We note that, in principle, solutions for a half-plane crack can be obtained from the ones for a *circular* crack by the limiting procedure, when the crack radius $\to \infty$. However, the implementation of this procedure involves evaluations of indeterminate ratios, some of which are very difficult. Although these difficulties can, in principle, be overcome (as outlined in Appendix A on a simple example), it is more straightforward to analyze the half-plane crack configuration as an independent problem.

In the expressions given in this work for displacements at point (x, y, z) and for SIFs due to the stress sources applied at (x, y, z), it is assumed that $z \geqslant 0$ (upper half-plane). Results for $z < 0$ follow from the symmetry relations $f(x, y, -z) = f(-x, -y, z)$ where f denotes any of the mentioned quantities (Kachanov (1993)).

In the earlier literature on the half-plane crack, the most fundamental contribution is due to Uflyand (1965), who used the integral transform techniques and obtained the potential functions for a pair of normal and tangential point forces applied at the crack faces. He considered both symmetric and antisymmetric arrangements of the point forces, when the forces are either normal or tangential (and normal to the crack edge) to the crack plane. However, his solutions were limited to computation of stresses in the plane of the crack. In the case of symmetric and antisymmetric shear point forces parallel to the crack edge, only a general outline for finding the potential functions was discussed by him. Full stress and displacement fields due to a pair of equal and opposite *normal* point forces applied at the crack faces were derived by Fabrikant and Karapetian (1994).

The SIFs (all three modes) due to point forces applied at the crack faces immediately follow from Uflyand's work (1965). In the explicit form, they were given by Sih and Liebowitz (1968). Kassir and Sih (1975) presented a summary of these results, along with SIFs for the case when the forces are applied at the points ($x = 0, y = 0, \pm z$) above and below the crack edge. Rice (1985a) derived the mode I weight function, from which the expression for K_I due to a point force *normal* to the half-plane crack and applied at an arbitrary point in space immediately follows. (He also analyzed a more general configuration, with a perturbed crack front line.) Bueckner (1987) discussed, in the context of weight functions, the problem of finding SIFs along the edge of both circular and half-plane cracks, induced by an arbitrarily located and oriented point force, but actual solution was given only for the case of a pair of point forces at the crack faces (analyzed earlier by Uflyand (1965) and Kassir and Sih (1975)). All these results are recovered in the present work as special cases.

We also mention several results for somewhat related, but different problems. Rice (1985b) and Hanson (1990, 1992) considered three-dimensional interactions of a crack with a coplanar dislocation loop. Karihaloo and Huang (1989) considered the problem of half-plane crack interacting with volumetric distributions of shear transformation strains.

2. DISPLACEMENT FIELD DUE TO A PAIR OF EQUAL AND OPPOSITE POINT FORCES APPLIED AT THE FACES OF A HALF-PLANE CRACK (TRANSVERSELY ISOTROPIC AND ISOTROPIC SOLIDS)

The solution of this problem (Fig. 1) was given by Fabrikant *et al.* (1993, 1995) and is used as a starting point in our work. It is transformed here to a somewhat simpler form, as follows. The point forces are applied at $x_0 < 0$, y_0, $z_0 = \pm 0$ where, without loss of generality, we set $y_0 = 0$.

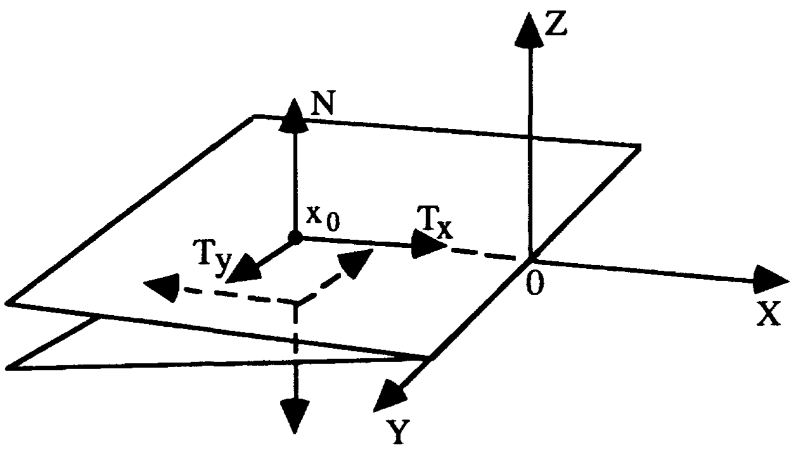

Fig. 1. The configuration of a half-plane crack loaded by a pair of equal and opposite point forces applied at crack faces.

A. *Transversely isotropic solid* (*crack plane is parallel to the isotropy plane* xy)

A-1. For a pair of *normal* point forces N, the displacement field is:

$$u_z = \frac{2}{\pi} NH \sum_{k=1}^{2} \frac{m_k}{m_k - 1} f_1(z_k) \tag{1}$$

$$u_x + iu_y = \frac{2}{\pi} NH \sum_{k=1}^{2} \frac{\gamma_k}{m_k - 1} f_2(z_k) \tag{2}$$

where γ_1, γ_2, γ_3, m_1, m_2 are the constants introduced by Elliott (1948). They can be expressed in terms of five transversely isotropic elastic constants A_{11}, A_{13}, A_{33}, A_{44} and A_{66} related to the "engineering constants" E_x, E_z, ν_{xz}, ν_{xy} and G_{xz} (Young's and shear moduli and Poisson's ratios):

$$\left.\begin{aligned} A_{11} &= \frac{E_x(E_z - \nu_{xz}^2 E_x)}{(1+\nu_{xy})(E_z - \nu_{xy}E_z - 2\nu_{xz}^2 E_x)}, \quad A_{13} = \frac{E_x E_z \nu_{xz}}{E_z - \nu_{xy}E_z - 2\nu_{xz}^2 E_x}, \\ A_{33} &= \frac{E_z^2(1-\nu_{xy})}{E_z - \nu_{xy}E_z - 2\nu_{xz}^2 E_x}, \quad A_{66} = \frac{E_x}{2(1+\nu_{xy})}, \quad A_{44} = G_{xz}, \end{aligned}\right\} \tag{3}$$

through the following equations: $\gamma_3 = (A_{44}/A_{66})^{1/2}$ and

$$\frac{A_{44} + m_k(A_{13} + A_{44})}{A_{11}} = \frac{m_k A_{33}}{m_k A_{44} + A_{13} + A_{44}} = \gamma_k^2, \quad k = 1, 2 \tag{4}$$

where the first equality sign represents a quadratic equation for m having complex conjugate roots m_1, m_2 (such that $m_1 m_2 = 1$) and the second equality sign determines γ_1 and γ_2 that can be shown to be complex conjugates ($\gamma_k = \sqrt{\gamma_k^2}$ are chosen in such a way that $\mathrm{Re}\,\gamma_k > 0$). The quantities z_k are defined as $z_k = z/\gamma_k$ ($k = 1, 2, 3$) so that z_1 and z_2 are complex conjugates and z_3 is real.

The following combinations of elastic constants are also used in the text to follow:

$$G_1 = \beta + \gamma_1\gamma_2 H, \quad G_2 = \beta - \gamma_1\gamma_2 H, \quad \beta = \gamma_3/2\pi A_{44}, \quad H = \frac{(\gamma_1+\gamma_2)A_{11}}{2\pi(A_{11}A_{33} - A_{13}^2)}. \tag{5}$$

A-2. For a pair of *tangential* point forces (to account for their direction, they are characterized by the complex quantity $T = T_x + iT_y$), we have

$$u_z = \frac{2}{\pi} H\gamma_1\gamma_2 \,\mathrm{Re}\left\{\sum_{k=1}^{2} \frac{m_k}{(m_k-1)\gamma_k}\left[\bar{f}_2(z_k) + \frac{G_2}{G_1}\bar{f}_3(z_k)\right]T\right\} \tag{6}$$

$$\begin{aligned} u_x + iu_y = {} & \frac{H\gamma_1\gamma_2}{\pi} \sum_{k=1}^{2} \frac{1}{m_k - 1}\left\{-\left[f_1(z_k) + \frac{G_2}{G_1}\bar{f}_4(z_k)\right]T + \left[f_5(z_k) + \frac{G_2}{G_1} f_6(z_k)\right]\bar{T}\right\} \\ & + \frac{\beta}{\pi}\left\{\left[f_1(z_3) - \frac{G_2}{G_1}\bar{f}_4(z_3)\right]T + \left[f_5(z_3) - \frac{G_2}{G_1} f_6(z_3)\right]\bar{T}\right\} \end{aligned} \tag{7}$$

where an overbar denotes a complex conjugate quantity, a function with an overbar $\bar{f}(\zeta)$ is understood as $\overline{f(\bar{\zeta})}$ and the functions $f_{1\text{-}6}(z_k) \equiv f_{1\text{-}6}(x, y, z_k; x_0)$ are defined as follows:

$$f_1 = \frac{1}{R_1}\arctan\left(\frac{h}{R_1}\right) \tag{8}$$

$$f_2 = \frac{1}{\bar{q}}\left[\sqrt{\frac{-2x_0}{\bar{s}}}\arctan\left(\frac{\bar{s}}{l_2}\right)^{1/2} - \frac{z}{R_1}\arctan\left(\frac{h}{R_1}\right)\right] \tag{9}$$

$$f_3 = \frac{\sqrt{-2x_0}}{s}\left[\frac{1}{\sqrt{s}}\arctan\left(\frac{s}{l_2}\right)^{1/2} - \frac{\sqrt{l_2}}{l_2+s}\right] \tag{10}$$

$$f_4 = \frac{h}{s}\left\{\frac{3}{s}\left[1-\left(\frac{l_2}{s}\right)^{1/2}\arctan\left(\frac{s}{l_2}\right)^{1/2}\right]-\frac{1}{l_2+s}\right\} \tag{11}$$

$$f_5 = \frac{R_1^2+z^2}{R_1\bar{q}^2}\arctan\left(\frac{h}{R_1}\right)+\frac{\sqrt{-2x_0}}{\bar{q}}\left[\frac{z}{\sqrt{\bar{s}}}\left(\frac{1}{\bar{s}}-\frac{2}{\bar{q}}\right)\arctan\left(\frac{\bar{s}}{l_2}\right)^{1/2} + \frac{1}{\sqrt{\bar{q}}}\arctan\left(\frac{\bar{q}}{l_1}\right)^{1/2} - \frac{\sqrt{l_1}}{\bar{s}}\right] \tag{12}$$

$$f_6 = \frac{\sqrt{-2x_0}}{\bar{q}}\left[\frac{1}{\sqrt{\bar{q}}}\arctan\left(\frac{\bar{q}}{l_1}\right)^{1/2} - \frac{\sqrt{l_1}}{l_2+s}\right] \tag{13}$$

B. *Isotropic solid*

The above given results simplify in this case to the following expressions.

B-1. For a pair of *normal* point forces N:

$$u_z = \frac{N(1+\nu)}{\pi^2 E}\left\{\left[\frac{2(1-\nu)}{R_1}+\frac{z^2}{R_1^3}\right]\arctan\left(\frac{h}{R_1}\right)+\frac{z^2}{R_1^2+h^2}\left[\frac{x_0}{h\sqrt{x^2+z^2}}+\frac{h}{R_1^2}\right]\right\} \tag{14}$$

$$u_x+iu_y = \frac{N(1+\nu)}{\pi^2 E}\left\{\frac{1-2\nu}{\bar{q}}\left[\frac{z}{R_1}\arctan\left(\frac{h}{R_1}\right)-\sqrt{\frac{-2x_0}{\bar{s}}}\arctan\left(\frac{\bar{s}}{l_2}\right)^{1/2}\right] + \frac{z}{\bar{q}}\left[\frac{h}{(l_1+l_2)(l_2+\bar{s})}+\frac{R_1^2-z^2}{R_1^3}\arctan\left(\frac{h}{R_1}\right)-\frac{z^2}{R_1^2+h^2}\left(\frac{x_0}{h\sqrt{x^2+z^2}}+\frac{h}{R_1^2}\right)\right]\right\}. \tag{15}$$

B-2. For a pair of *tangential* point forces T:

$$u_z = \frac{1+\nu}{\pi^2 E}\mathrm{Re}\left\{\left[(1-2\nu)\left(\bar{f}_2(z)+\frac{\nu}{2-\nu}\bar{f}_3(z)\right)-z f_7(z)\right]T\right\} \tag{16}$$

$$u_x+iu_y = \frac{1+\nu}{\pi^2 E}\left\{\left[(2-\nu)f_1(z)-\frac{\nu^2}{2-\nu}\bar{f}_4(z)\right]T+f_8(z)\bar{T}+\frac{z}{2}[f_9(z)T-f_{10}(z)\bar{T}]\right\} \tag{17}$$

where the functions $f_{7\text{–}10}(z) \equiv f_{7\text{–}10}(x, y, z; x_0)$ are defined as follows:

$$f_7 = \frac{1}{\bar{q}}\left\{\frac{z^2}{R_1^2+h^2}\left[\frac{x_0}{h\sqrt{x^2+z^2}}+\frac{h}{R_1^2}\right]-\frac{R_1^2-z^2}{R_1^3}\arctan\left(\frac{h}{R_1}\right)-\frac{h}{(l_1+l_2)(l_2+s)}\right\} - \frac{\nu}{2-\nu}\frac{h}{(l_2+\bar{s})^2\sqrt{x^2+z^2}} \tag{18}$$

$$f_8 = \frac{\nu}{\bar{q}}\left\{\frac{R_1^2+z^2}{R_1\bar{q}}\arctan\left(\frac{h}{R_1}\right)+\sqrt{-2x_0}\left[\frac{z}{\sqrt{\bar{s}}}\left(\frac{1}{\bar{s}}-\frac{2}{\bar{q}}\right)\arctan\left(\frac{\bar{s}}{l_2}\right)^{1/2}+\frac{\sqrt{l_1}}{l_2+s}\right]-\frac{h}{\bar{s}}\right\} \tag{19}$$

$$f_9 = \frac{\nu}{2-\nu}\left\{\frac{z}{\bar{s}\sqrt{x^2+z^2}}\left[\frac{1}{l_2+\bar{s}}\left(\frac{x_0}{h}+\frac{h}{l_2+\bar{s}}\right)-\frac{3}{\bar{s}}\left(\frac{x_0}{h}-\frac{h}{2(l_2+\bar{s})}\right)\right]\right.$$
$$\left.-\frac{3\sqrt{-2x_0}}{\bar{s}^{5/2}}\arctan\left(\frac{\bar{s}}{l_2}\right)^{1/2}\right\}-\frac{z}{R_1^3}\arctan\left(\frac{h}{R_1}\right)+\frac{z}{R_1^2+h^2}\left(\frac{x_0}{h\sqrt{x^2+z^2}}+\frac{h}{R_1^2}\right) \tag{20}$$

$$f_{10} = \frac{1}{\bar{q}}\left\{\frac{z(3R_1^2-z^2)}{R_1^3\bar{q}}\arctan\left(\frac{h}{R_1}\right)-\frac{z(R_1^2+z^2)}{\bar{q}(R_1^2+h^2)}\left(\frac{x_0}{h\sqrt{x^2+z^2}}+\frac{h}{R_1^2}\right)\right.$$
$$+\sqrt{\frac{-2x_0}{l_2}}\left[\left(\frac{1}{\bar{s}}-\frac{2}{\bar{q}}\right)\left(\left(\frac{l_2}{\bar{s}}\right)^{1/2}\arctan\left(\frac{\bar{s}}{l_2}\right)^{1/2}-\frac{z^2}{(l_1+l_2)(l_2+\bar{s})}\right)\right]$$
$$\left.+\frac{x_0 z}{h\sqrt{x^2+z^2}}\left[\frac{1}{\bar{s}}+\frac{2}{2-\nu}\frac{1}{l_1+\bar{q}}+\frac{\nu}{2-\nu}\left(\frac{1}{l_2+s}-\frac{2l_1}{(l_2+s)^2}\right)\right]\right\} \tag{21}$$

The following notations are used in all the formulas above:

$$R_1 = \sqrt{(x-x_0)^2+y^2+z^2},\quad s = -(x+x_0)-iy,\quad q = x-x_0+iy$$
$$h = \sqrt{-2x_0[(x^2+z^2)^{1/2}-x]},\quad l_1 = \sqrt{x^2+z^2}-x,\quad l_2 = \sqrt{x^2+z^2}+x. \tag{22}$$

3. INTERACTION OF A HALF-PLANE CRACK WITH A POINT FORCE (TRANSVERSELY ISOTROPIC AND ISOTROPIC SOLIDS)

We derive SIFs along the edge of a half-plane crack due to a point force applied at an arbitrary point and having an arbitrary direction. We use the reciprocity theorem that relates the displacements at some point (x, y, z) due to a pair of equal and opposite point forces applied at the point $(x_0, y_0 = 0, z_0 \pm 0)$ of the crack faces (given in Section 2) to the displacement discontinuity at this point of the crack due to a point force applied at (x, y, z). We will also need representations of SIFs in terms of the normal displacement discontinuity $[u_z]$ and the tangential displacement discontinuity in the complex form $\Delta = [u_x]+i[u_y]$. A similar representation was derived for *circular* crack of radius a by Fabrikant (1989):

$$K_I = \frac{1}{8\pi H}\lim_{\rho_0\to a}\frac{[u_z]}{\sqrt{a-\rho_0}},\quad K_{II}+iK_{III} = \frac{a}{2\pi(G_2^2-G_1^2)\sqrt{2a}}\lim_{\rho_0\to a}\left[\frac{G_1\,e^{-i\phi_0}\Delta+G_2\,e^{i\phi_0}\bar{\Delta}}{(a^2-\rho_0^2)^{1/2}}\right] \tag{23}$$

where Cartesian coordinates x, y (with the origin at the crack center) are oriented with respect to the polar coordinates ρ_0, ϕ_0 of a point on the crack in such a way that ϕ_0 is counted counterclockwise from the x-axis.

The representation of this kind for a *half-plane* crack can be obtained by calculating the limit of (23) as $a \to \infty$. This yields:

$$K_I = \frac{1}{8\pi H}\lim_{x_0\to 0}\frac{[u_z]}{\sqrt{-x_0}},\quad K_{II}+iK_{III} = \frac{1}{4\pi(G_2^2-G_1^2)}\lim_{x_0\to 0}\left(\frac{G_1\Delta+G_2\bar{\Delta}}{\sqrt{-x_0}}\right) \tag{24}$$

where x and y in the expression $\Delta = [u_x]+i[u_y]$ denote now the axes normal to and along

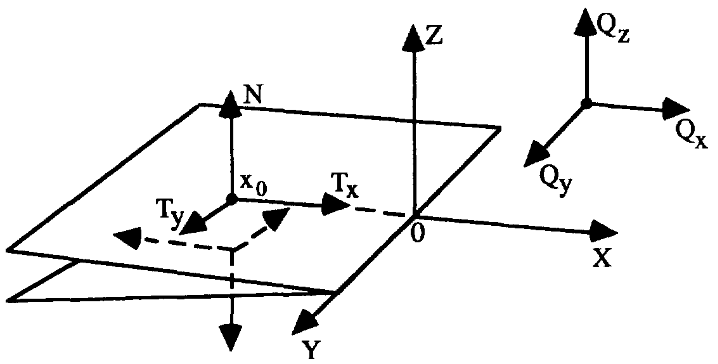

Fig. 2. Dual systems of forces (Q_x, Q_y, Q_z) and (T_x, T_y, N) used in the reciprocity theorem.

the crack edge, respectively. Results for the *isotropic* solid are obtained by setting $H = (1-\nu^2)/(\pi E)$, $G_1 = (2-\nu)(1+\nu)/(\pi E)$, $G_2 = \nu(1+\nu)/(\pi E)$ in (24).

In the text to follow, we calculate the right hand part of (24), i.e. quantities $[u_z]$, Δ and the limits.

A. *Transversely isotropic solid*

The derivation is based on considering two dual systems of loads (Fig. 2) and using the reciprocity theorem. We first consider the systems of loads related to the *opening mode* displacements and derive the expressions for K_I.

In the first loading system, a point force Q_z is applied at an arbitrary point (x, y, z) in the positive z-direction, producing a normal displacement discontinuity $[u_z^{Q_z}]$ (to be found). In the second loading system, two equal and opposite point forces N normal to the crack are applied at the points $(x_0, 0, \pm 0)$ of the crack, producing z-displacement component u_z^N at the point (x, y, z) (given by (1)). The reciprocity theorem states that

$$N[u_z^{Q_z}] = Q_z u_z^N \tag{25}$$

yielding

$$[u_z^{Q_z}] = \frac{2}{\pi} Q_z H \left[\sum_{k=1}^{2} \frac{m_k}{m_k - 1} f_1(z_k) \right] \tag{26}$$

where the function $f_1(z_k)$ is defined by (8).

Normal displacement discontinuity $[u_z^Q]$ due to a point force (Q_x, Q_y) applied at (x, y, z) in the direction parallel to the crack can be found in a similar way, by considering a dual problem: a pair of two equal and opposite normal point forces N is applied at the crack faces at the points $(x_0, 0, \pm 0)$ and produces the displacement components u_x^N, u_y^N at the point (x, y, z) (given by (2)). This yields

$$[u_z^{Q_x}] = \frac{2}{\pi} Q_x H \,\mathrm{Re}\left[\sum_{k=1}^{2} \frac{\gamma_k}{m_k - 1} f_2(z_k) \right], \tag{27}$$

$$[u_z^{Q_y}] = \frac{2}{\pi} Q_y H \,\mathrm{Im}\left[\sum_{k=1}^{2} \frac{\gamma_k}{m_k - 1} f_2(z_k) \right], \tag{28}$$

with $f_2(z_k)$ defined by (9).

Substituting (26)–(28) into (24_I) and calculating the limits yields K_I due to Q_z, Q_x, Q_y:

$$K_{\mathrm{I}}(y) = \begin{Bmatrix} Q_x \\ Q_y \\ Q_z \end{Bmatrix} \frac{1}{2\pi^2\sqrt{2}} \begin{Bmatrix} \operatorname{Re} \sum_{k=1}^{2} \frac{\gamma_k}{m_k - 1} g_2(z_k) \\ \operatorname{Im} \sum_{k=1}^{2} \frac{\gamma_k}{m_k - 1} g_2(z_k) \\ \sum_{k=1}^{2} \frac{m_k}{m_k - 1} g_1(z_k) \end{Bmatrix} \tag{29}$$

where two elementary functions $g_{1-2}(x, y, z_k)$ are introduced:

$$g_1 = \frac{\sqrt{l_1}}{R^2}, \quad g_2 = \frac{1}{\bar{t}}\left[\frac{z\sqrt{l_1}}{R^2} - \frac{1}{\sqrt{\bar{t}}}\arctan\left(\frac{\bar{t}}{l_2}\right)^{1/2}\right] \tag{30}$$

and the following notations are used:

$$t = -x - iy, \quad R^2 = x^2 + y^2 + z^2. \tag{31}$$

We now consider the systems of reciprocal loads related to the *tangential* displacement discontinuity $\Delta = [u_x] + i[u_y]$ and derive Δ and K_{II}, K_{III} due to a point force (Q_x, Q_y, Q_z) applied at an arbitrary point (x, y, z). In contrast with straightforward calculations for the mode I, this case involves some mathematical subtleties discussed in Appendix B. The results are as follows.

The discontinuity Δ due to a point force Q_z and to a point force (Q_x, Q_y), both applied at the point (x, y, z), are:

$$\Delta^{Q_z} = \frac{2}{\pi} Q_z H\gamma_1\gamma_2 \sum_{k=1}^{2} \frac{\bar{m}_k}{(\bar{m}_k - 1)\bar{\gamma}_k}\left[f_2(z_k) + \frac{G_2}{G_1} f_3(z_k)\right] \tag{32}$$

$$\begin{aligned}\Delta^{Q_x} + \Delta^{Q_y} = \frac{H\gamma_1\gamma_2}{\pi} \sum_{k=1}^{2} \frac{1}{\bar{m}_k - 1}\Bigg\{ &-\left[\bar{f}_1(z_k) + \frac{G_2}{G_1} f_4(z_k)\right](Q_x + iQ_y) \\ &+ \left[f_5(z_k) + \frac{G_2}{G_1} f_6(z_k)\right](Q_x - iQ_y)\Bigg\} + \frac{\beta}{\pi}\Bigg\{\left[f_1(z_3) - \frac{G_2}{G_1} f_4(z_3)\right](Q_x + iQ_y) \\ &+ \left[f_5(z_3) - \frac{G_2}{G_1} f_6(z_3)\right](Q_x - iQ_y)\Bigg\}\end{aligned} \tag{33}$$

where the function f_{1-6} are defined by (8)–(13).

In order to find mode II and III SIFs due to (Q_x, Q_y, Q_z), we substitute (32) and (33) into (24_2) and calculate the limit. This yields:

$$K_{\mathrm{II}}(y) = \begin{Bmatrix} Q_x \\ Q_y \\ Q_z \end{Bmatrix} \frac{1}{4\pi^2\sqrt{2}}$$

$$\times \begin{Bmatrix} \operatorname{Re}\left\{\sum_{k=1}^{2} \frac{1}{m_k - 1}\left[\left(\bar{g}_1(z_k) + \frac{G_2}{G_1} g_4(z_k)\right) - \left(g_5(z_k) + \frac{G_2}{G_1} g_6(z_k)\right)\right] \right. \\ \left. - \frac{G_1 + G_2}{G_1 - G_2}\left[\left(g_1(z_3) - \frac{G_2}{G_1} g_4(z_3)\right) + \left(g_5(z_3) - \frac{G_2}{G_1} g_6(z_3)\right)\right]\right\}. \\ \operatorname{Im}\left\{\sum_{k=1}^{2} \frac{1}{m_k - 1}\left[-\left(\bar{g}_1(z_k) + \frac{G_2}{G_1} g_4(z_k)\right) - \left(g_5(z_k) + \frac{G_2}{G_1} g_6(z_k)\right)\right] \right. \\ \left. + \frac{G_1 + G_2}{G_1 - G_2}\left[\left(g_1(z_3) - \frac{G_2}{G_1} g_4(z_3)\right) - \left(g_5(z_3) - \frac{G_2}{G_1} g_6(z_3)\right)\right]\right\}. \\ \operatorname{Re}\left\{\sum_{k=1}^{2} \frac{-2m_k}{(m_k - 1)\gamma_k}\left[\bar{g}_2(z_k) + \frac{G_2}{G_1} \bar{g}_3(z_k)\right]\right\} \end{Bmatrix} \tag{34}$$

$$K_{\mathrm{III}}(y)=\begin{Bmatrix}Q_x\\Q_y\\Q_z\end{Bmatrix}\frac{1}{4\pi^2\sqrt{2}}$$

$$\times\begin{Bmatrix}\mathrm{Im}\left\{\frac{G_1-G_2}{G_1+G_2}\sum_{k=1}^{2}\frac{1}{m_k-1}\left[\left(\bar{g}_1(z_k)+\frac{G_2}{G_1}g_4(z_k)\right)-\left(g_5(z_k)+\frac{G_2}{G_1}g_6(z_k)\right)\right]\right.\\ \left.-\left[\left(g_1(z_3)-\frac{G_2}{G_1}g_4(z_3)\right)+\left(g_5(z_3)-\frac{G_2}{G_1}g_6(z_3)\right)\right]\right\}.\\ \mathrm{Re}\left\{\frac{G_1-G_2}{G_1+G_2}\sum_{k=1}^{2}\frac{1}{m_k-1}\left[\left(\bar{g}_1(z_k)+\frac{G_2}{G_1}g_4(z_k)\right)+\left(g_5(z_k)+\frac{G_2}{G_1}g_6(z_k)\right)\right]\right.\\ \left.-\left[\left(g_1(z_3)-\frac{G_2}{G_1}g_4(z_3)\right)-\left(g_5(z_3)-\frac{G_2}{G_1}g_6(z_3)\right)\right]\right\}.\\ \mathrm{Im}\left\{\frac{G_1-G_2}{G_1+G_2}\sum_{k=1}^{2}\frac{2m_k}{(m_k-1)\gamma_k}\left[\bar{g}_2(z_k)+\frac{G_2}{G_1}\bar{g}_3(z_k)\right]\right\}\end{Bmatrix} \quad (35)$$

where the functions $g_{3\text{-}6}(x,y,z_k)$ are introduced:

$$g_3=\frac{1}{t}\left[\frac{1}{\sqrt{t}}\arctan\left(\frac{t}{l_2}\right)^{1/2}-\frac{\sqrt{l_2}}{l_2+t}\right] \quad (36)$$

$$g_4=\frac{\sqrt{l_1}}{t}\left\{\frac{3}{t}\left[1-\left(\frac{l_2}{t}\right)^{1/2}\arctan\left(\frac{t}{l_2}\right)^{1/2}\right]-\frac{1}{l_2+t}\right\} \quad (37)$$

$$g_5=\frac{(R^2+z^2)\sqrt{l_1}}{R^2\bar{t}^2}-\frac{1}{\bar{t}}\left[\frac{3z}{\bar{t}^{3/2}}\arctan\left(\frac{\bar{t}}{l_2}\right)^{1/2}+\frac{1}{\sqrt{-\bar{t}}}\arctan\left(-\frac{\bar{t}}{l_1}\right)^{1/2}-\frac{\sqrt{l_1}}{\bar{t}}\right] \quad (38)$$

$$g_6=-\frac{1}{\bar{t}}\left[\frac{1}{\sqrt{-\bar{t}}}\arctan\left(-\frac{\bar{t}}{l_1}\right)^{1/2}-\frac{\sqrt{l_1}}{l_2+t}\right] \quad (39)$$

and notations (31) and (22) are used.

B. *Isotropic solid*

Results for the isotropic solid can, in principle, be obtained from those for the transversely isotropic solid. This, however, requires calculation of non-trivial limits. Alternatively, the case of the isotropic solid can be analyzed independently, following the same procedure as the one for the transversely isotropic case and using (14)–(17) and (24). The results are as follows.

For the *normal* displacement discontinuity:

$$[u_z^{Q_z}]=\frac{Q_z(1+v)}{\pi^2E}\left\{\left[\frac{2(1-v)}{R_1}+\frac{z^2}{R_1^3}\right]\arctan\left(\frac{h}{R_1}\right)+\frac{z^2}{R_1^2+h^2}\left[\frac{x_0}{h\sqrt{x^2+z^2}}+\frac{h}{R_1^2}\right]\right\} \quad (40)$$

$$[u_z^{Q_x}]=\frac{Q_x(1+v)}{\pi^2E}\mathrm{Re}\left\{\frac{1-2v}{\bar{q}}\left[\frac{z}{R_1}\arctan\left(\frac{h}{R_1}\right)-\sqrt{\frac{-2x_0}{\bar{s}}}\arctan\left(\frac{\bar{s}}{l_2}\right)^{1/2}\right]\right.$$
$$\left.+\frac{z}{\bar{q}}\left[\frac{h}{(l_1+l_2)(l_2+\bar{s})}+\frac{R_1^2-z^2}{R_1^3}\arctan\left(\frac{h}{R_1}\right)-\frac{z^2}{R_1^2+h^2}\left(\frac{x_0}{h\sqrt{x^2+z^2}}+\frac{h}{R_1^2}\right)\right]\right\}. \quad (41)$$

$$[u_z^{Q_y}] = \frac{Q_y(1+\nu)}{\pi^2 E}\operatorname{Im}\left\{\frac{1-2\nu}{\bar{q}}\left[\frac{z}{R_1}\arctan\left(\frac{h}{R_1}\right)-\sqrt{\frac{-2x_0}{\bar{s}}}\arctan\left(\frac{\bar{s}}{l_2}\right)^{1/2}\right]\right.$$
$$\left.+\frac{z}{\bar{q}}\left[\frac{h}{(l_1+l_2)(l_2+\bar{s})}+\frac{R_1^2-z^2}{R_1^3}\arctan\left(\frac{h}{R_1}\right)-\frac{z^2}{R_1^2+h^2}\left(\frac{x_0}{h\sqrt{x^2+z^2}}+\frac{h}{R_1^2}\right)\right]\right\}. \quad (42)$$

For the *tangential* displacement discontinuities:

$$\Delta^{Q_z} = \frac{Q_z(1+\nu)}{\pi^2 E}\left\{(1-2\nu)\left[f_2(z)+\frac{\nu}{2-\nu}f_3(z)\right]-z\bar{f}_7(z)\right\} \quad (43)$$

$$\Delta^{Q_x}+\Delta^{Q_y} = \frac{1+\nu}{\pi^2 E}\left\{\left[(2-\nu)f_1(z)-\frac{\nu^2}{2-\nu}f_4(z)\right](Q_x+iQ_y)+f_8(z)(Q_x-iQ_y)\right.$$
$$\left.+\frac{z}{2}[\bar{f}_9(z)(Q_x+iQ_y)-f_{10}(z)(Q_x-iQ_y)]\right\} \quad (44)$$

where all $f(z)$ functions are defined in Section 2.

The SIFs for all three modes are:

$$K_{\rm I}(y) = \begin{Bmatrix} Q_x \\ Q_y \\ Q_z \end{Bmatrix}\frac{\sqrt{l_1}}{4\pi^2(1-\nu)\sqrt{2}}\begin{Bmatrix} \operatorname{Re} g_8 \\ \operatorname{Im} g_8 \\ g_7 \end{Bmatrix} \quad (45)$$

$$K_{\rm II}(y) = \begin{Bmatrix} Q_x \\ Q_y \\ Q_z \end{Bmatrix}\frac{\sqrt{l_1}}{4\pi^2(1-\nu)\sqrt{2}}\operatorname{Re}\begin{Bmatrix} g_{10}-g_{11} \\ i(g_{10}+g_{11}) \\ g_9 \end{Bmatrix} \quad (46)$$

$$K_{\rm III}(y) = \begin{Bmatrix} Q_x \\ Q_y \\ Q_z \end{Bmatrix}\frac{\sqrt{l_1}}{4\pi^2\sqrt{2}}\operatorname{Im}\begin{Bmatrix} g_{10}-g_{11} \\ i(g_{10}+g_{11}) \\ g_9 \end{Bmatrix} \quad (47)$$

where the functions $g_{7\text{–}11}(x,y,z)$ are introduced:

$$g_7 = \frac{1}{R^2}\left[2(1-\nu)+\frac{2z^2}{R^2}-\frac{l_2}{l_1+l_2}\right] \quad (48)$$

$$g_8 = -\frac{1}{\bar{t}}\left\{\frac{z}{R^2}\left[2(1-\nu)-\frac{2z^2}{R^2}+\frac{l_2}{l_1+l_2}\right]+\frac{z}{(l_1+l_2)(l_2+\bar{t})}-(1-2\nu)\frac{1}{\sqrt{\bar{t}l_1}}\arctan\left(\frac{\bar{t}}{l_2}\right)^{1/2}\right\}$$
$$(49)$$

$$g_9 = \frac{1}{\bar{t}}\left[\frac{z}{R^2}\left(2\nu-\frac{2z^2}{R^2}+\frac{l_2}{l_1+l_2}\right)+\frac{z}{(l_1+l_2)(l_2+\bar{t})}+(1-2\nu)\frac{1}{\sqrt{\bar{t}l_1}}\arctan\left(\frac{\bar{t}}{l_2}\right)^{1/2}\right]$$
$$+\frac{\nu(1-2\nu)}{2-\nu}\frac{1}{t}\left[\frac{z}{l_1(l_2+t)}-\frac{1}{\sqrt{tl_1}}\arctan\left(\frac{t}{l_2}\right)^{1/2}\right]-\frac{\nu}{2-\nu}\frac{2z}{(l_1+l_2)(l_2+t)^2} \quad (50)$$

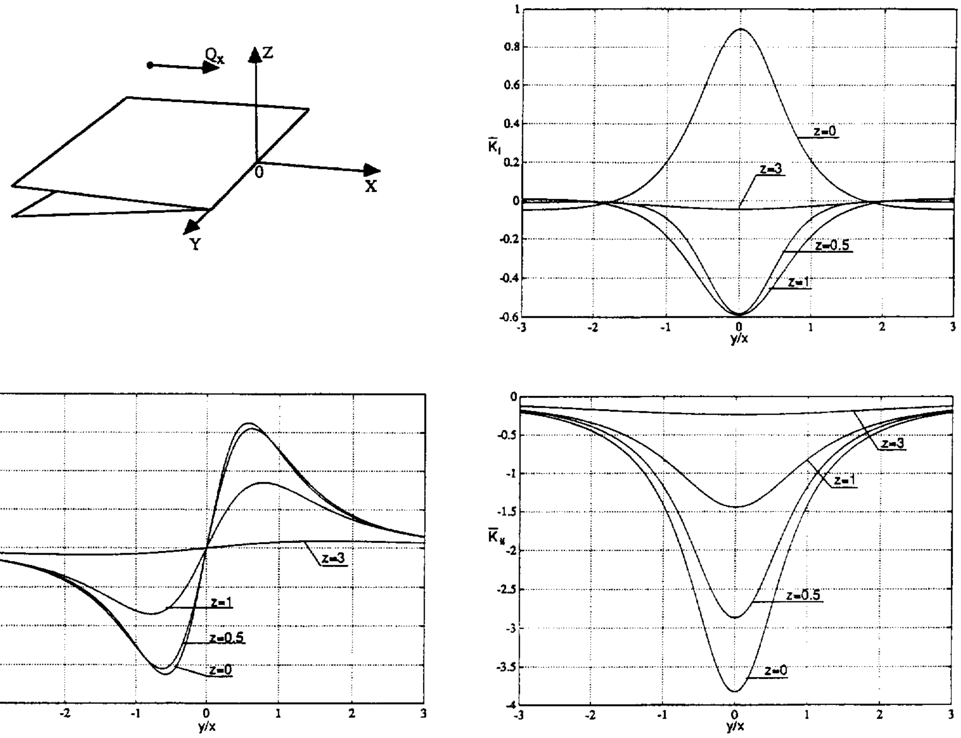

Fig. 3. Dimensionless stress intensity factors $\bar{K}_{(\mathrm{I,II,III})} = 4\pi^2\sqrt{2}x^{3/2}Q_x^{-1}K_{(\mathrm{I,II,III})}$ along the crack edge due to a concentrated force Q_x applied at the point ($x=-1; y=0; z$) for several values of z. Poisson's ratio $\nu = 0.3$.

$$g_{10} = \frac{1}{2R^2}\left[\frac{2z^2}{R^2} - \frac{l_2}{l_1+l_2} - 2(2-\nu)\right] + \frac{\nu}{2-\nu}\frac{1}{2t}\left[\left(\frac{3}{t} - \frac{1}{l_2+t}\right)\left(2\nu - \frac{l_2}{l_1+l_2} - \frac{z^2}{(l_1+l_2)(l_2+t)}\right)\right.$$
$$\left. - \frac{3z^2}{(l_1+l_2)(l_2+t)^2} + (1-2\nu)\frac{3}{t}\left(\frac{l_2}{t}\right)^{1/2}\arctan\left(\frac{t}{l_2}\right)^{1/2}\right] \quad (51)$$

$$g_{11} = \frac{1}{2\bar{t}^2}\left\{\frac{R^2+z^2}{R^2}\left(2\nu - \frac{l_2}{l_1+l_2}\right) + 2\nu - \frac{2z^2(R^2-z^2)}{R^4} - \frac{3z^2}{(l_1+l_2)(l_2+\bar{t})}\right.$$
$$+ 3(1-2\nu)\left(\frac{l_2}{\bar{t}}\right)^{1/2}\arctan\left(\frac{\bar{t}}{l_2}\right)^{1/2} - \frac{l_2}{l_1+l_2}\left(1 + \frac{2}{2-\nu}\frac{\bar{t}}{l_1-\bar{t}}\right)$$
$$\left. - \frac{\bar{t}}{l_2+t}\left[2\nu + \frac{\nu}{2-\nu}\left(\frac{l_2}{l_1+l_2} - \frac{2z^2}{(l_1+l_2)(l_2+t)}\right)\right]\right\} \quad (52)$$

and l_1, l_2, t and R are defined by (22) and (31).

The results (45)–(47) for SIFs are illustrated in Figs 3, 4 and 5.

An interesting observation, that may seem counterintuitive, can be made. If the point of application of Q is in the plane of the crack (outside of the crack), then, as follows from (45)–(47), the Q_z (or Q_x, Q_y) component of Q does not generate any mode I (or mode II, III) SIFs. This is best explained via the reciprocity theorem. By symmetry, mode I crack opening displacement does not generate any u_z displacement in the plane coplanar to the crack. Therefore, Q_z produces no normal crack opening displacement and, hence, no K_I. The result related to the tangential to the crack can be interpreted in a similar way.

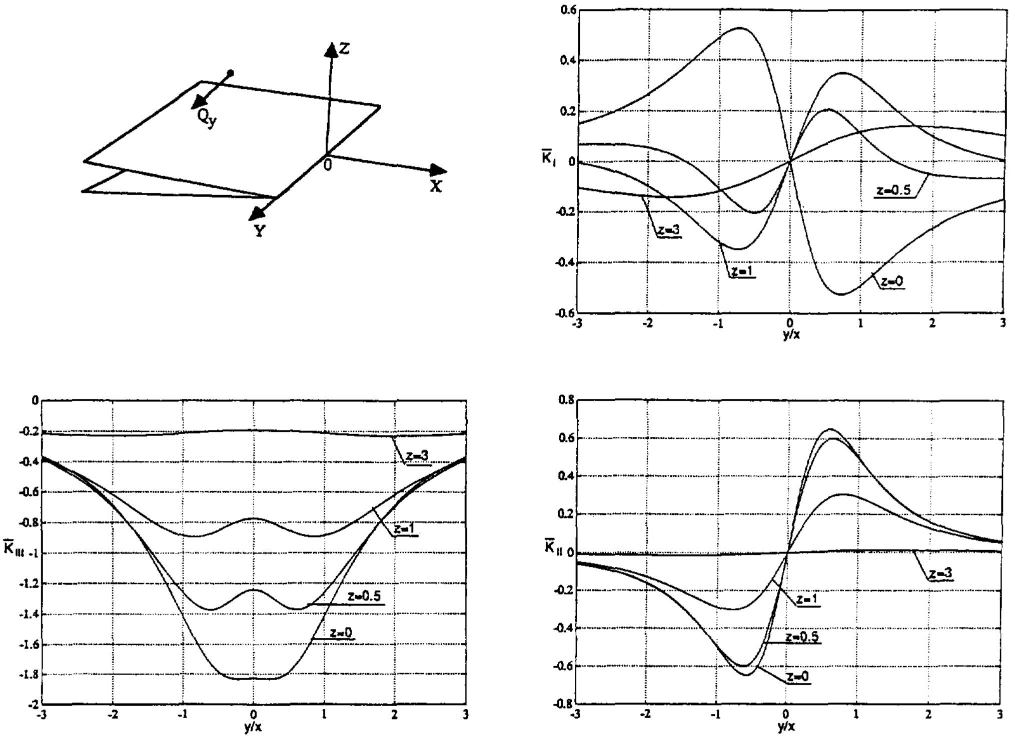

Fig. 4. Dimensionless stress intensity factors $\bar{K}_{(I,II,III)} = 4\pi^2\sqrt{2}x^{3/2}Q_y^{-1}K_{(I,II,III)}$ along the crack edge due to a concentrated force Q_y applied at the point $x = -1; y = 0; z)$ for several values of z. Poisson's ratio $\nu = 0.3$.

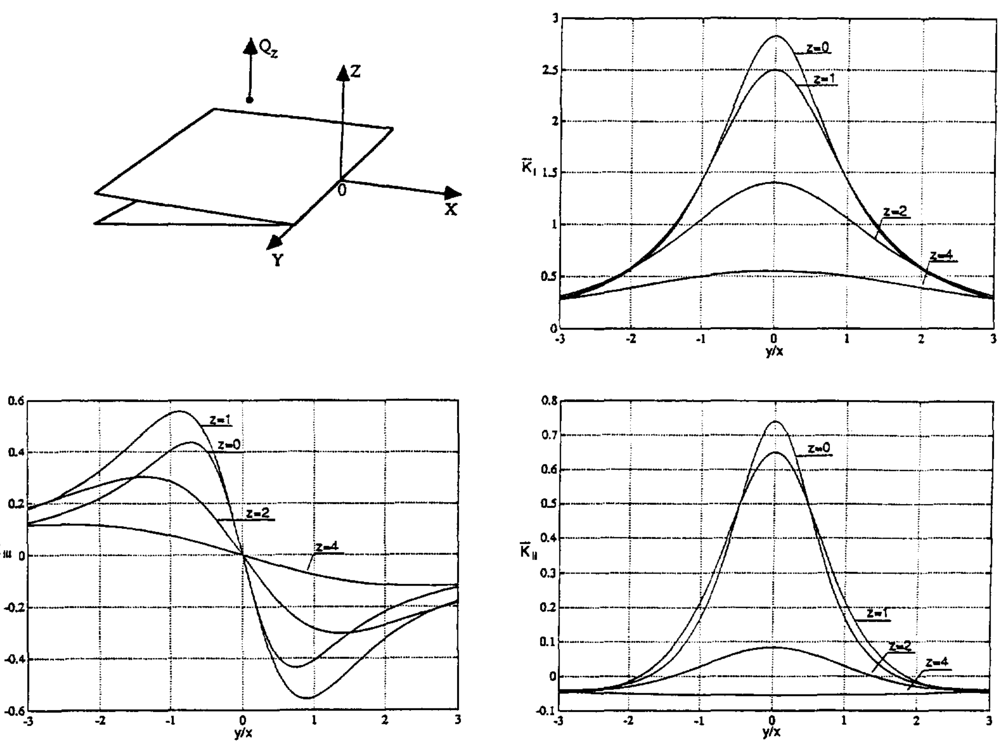

Fig. 5. Dimensionless stress intensity factors $\bar{K}_{(I,II,III)} = 4\pi^2\sqrt{2}x^{3/2}Q_z^{-1}K_{(I,II,III)}$ along the crack edge due to a concentrated force Q_z applied at the point $(x = -1; y = 0; z)$ for several values of z. Poisson's ratio $\nu = 0.3$.

4. SPECIAL CASES

Several special cases are considered here: the important case when tractions are applied at the crack faces and the case of point forces applied above the crack front line. For these configurations, solutions are partially available from the earlier literature (for the symmetric arrangements of two equal and opposite point forces). We recover these solutions and cover the more general asymmetric case when only one point force is applied.

A. *A point force* (Q_x, Q_y, Q_x) *is applied at one of the crack faces* $(z = 0, x < 0)$

The results for SIFs are:

$$\left.\begin{matrix} K_{\rm I}(y) \\ K_{\rm II}(y) \\ K_{\rm III}(y) \end{matrix}\right\} = \left\{\begin{matrix} \frac{1-2\nu}{8\pi\sqrt{2}(2-\nu)}\left[Q_x \operatorname{Re}\left(\frac{1}{-x+iy}\right)^{3/2} + Q_y \operatorname{Im}\left(\frac{1}{-x+iy}\right)^{3/2}\right] + Q_z \frac{1}{2\pi^2}\frac{\sqrt{-x}}{x^2+y^2}. \\ \frac{1}{2\pi^2}\frac{-\sqrt{-x}}{x^2+y^2}\left[Q_x\left(1+\frac{2\nu}{2-\nu}\frac{x^2-y^2}{x^2+y^2}\right)+Q_y\frac{4\nu}{2-\nu}\frac{xy}{x^2+y^2}\right] \\ +Q_z\frac{1-2\nu}{4\pi\sqrt{2}(2-\nu)}\operatorname{Re}\left(\frac{1}{-x+iy}\right)^{3/2}. \\ \frac{1}{2\pi^2}\frac{-\sqrt{-x}}{x^2+y^2}\left[Q_x\frac{4\nu}{2-\nu}\frac{xy}{x^2+y^2}+Q_y\left(1-\frac{2\nu}{2-\nu}\frac{x^2-y^2}{x^2+y^2}\right)\right] \\ +Q_z\frac{1-2\nu}{4\pi\sqrt{2}(2-\nu)}\operatorname{Im}\left(\frac{1}{-x+iy}\right)^{3/2}. \end{matrix}\right\}. \tag{53}$$

B. *Two equal and opposite point forces are applied at crack faces* $(z = \pm 0, x < 0)$

In this case, the results (53) simplify to the following expressions:

$$\left.\begin{matrix} K_{\rm I}(y) \\ K_{\rm II}(y) \\ K_{\rm III}(y) \end{matrix}\right\} = \frac{-\sqrt{-x}}{\pi^2(x^2+y^2)}\left\{\begin{matrix} -Q_z \\ Q_x\left(1+\frac{2\nu}{2-\nu}\frac{x^2-y^2}{x^2+y^2}\right)+Q_y\frac{2\nu}{2-\nu}\frac{2xy}{x^2+y^2} \\ Q_x\frac{2\nu}{2-\nu}\frac{2xy}{x^2+y^2}+Q_y\left(1-\frac{2\nu}{2-\nu}\frac{x^2-y^2}{x^2+y^2}\right) \end{matrix}\right\}. \tag{54}$$

The expressions (54) recover the results of Kassir and Sih (1975) and Bueckner (1987).

Note an interesting feature of the solution (54). At $\nu < 2/7$, the value of $|K_{\rm III}|$ due to Q_y is maximal at the point $y = 0$ (as intuitively expected). However, at $\nu > 2/7$, the maximum of $|K_{\rm III}|$ is reached at two points of the crack front (symmetrically located with respect to $y = 0$) and it substantially exceeds the value of $|K_{\rm III}|$ at $y = 0$ (the curve $z = 0$ in Fig. 4 corresponds to $\nu = 0.3$, close to the transitional point 2/7, and, therefore, has a plateau-like maximum).

Another interesting observation is that $K_{\rm II}$ due to Q_y is very close to $K_{\rm III}$ due to Q_x (they coincide exactly when the point of application of the force is $z = 0$).

C. *Two equal and opposite point forces* Q_z *are applied above and below the crack edge at the points* $(x = 0, \pm z)$

The results drastically simplify to the following expressions:

$$K_{\rm I}(y) = \frac{Q_z\sqrt{z}}{4\pi^2(1-\nu)\sqrt{2}}\frac{1}{z^2+y^2}\left(7-4\nu-\frac{4y^2}{z^2+y^2}\right), \quad K_{\rm II} = K_{\rm III} = 0; \tag{55}$$

This formula recovers the result of Kassir and Sih (1975).

5. INTERACTION OF A HALF-PLANE CRACK WITH DIPOLES

From now on, the solid is assumed *isotropic*. This assumption is made in order to avoid making formulas, that are already lengthy, even lengthier. The case of the transversely isotropic solid (with the crack parallel to the plane of isotropy) can be analyzed similarly.

A system of two equal and opposite point forces $\mathbf{Q} = (Q_x, Q_y, Q_z)$ and $-\mathbf{Q}$ with points of application lying on the line of $\mathbf{Q}$ and separated by distance h, in the limit of $Q \to \infty$ and $h \to 0$, is called a dipole. The limiting value of the product $\lim Qh \equiv P$ is called dipole's intensity (assumed to be finite); it is taken to be positive (negative) if the forces are directed away from (towards) each other.

The solution of K_{I} can be obtained from that for the point force by taking the directional derivative of (45) in the direction of $\mathbf{Q}$. Since $Q_k = Q\alpha_k$ $(k = x, y, z)$, where α_k are directional cosines between the x, y, z axes and the dipole line, we have:

$$K_{\mathrm{I}}(y) = \frac{P}{4\pi^2(1-\nu)\sqrt{2}} \sum_{m,n=x,y,z} \alpha_n \alpha_m \frac{\partial F_m}{\partial n} \tag{56}$$

where $F_x \equiv \sqrt{l_1}\,\mathrm{Re}\,g_8$, $F_y \equiv \sqrt{l_1}\,\mathrm{Im}\,g_8$, $F_z \equiv \sqrt{l_1}\,g_7$. Note that (56) is invariant with respect to the choice of direction along the dipole line (it is quadratic in α_k).

Calculations yield the following expression for K_{I}:

$$K_{\mathrm{I}}(y) = \frac{P\sqrt{l_1}}{4\pi^2(1-\nu)\sqrt{2}} \{\alpha_x\,\mathrm{Re}\,[\alpha_x p_1 + \alpha_y p_2 + \alpha_z p_3] + \alpha_y\,\mathrm{Im}\,[\alpha_x p_1 + \alpha_y p_2 + \alpha_z p_3] + \alpha_z[\alpha_x p_4 + \alpha_y p_5 + \alpha_z p_6]\} \tag{57}$$

where the elementary functions $p_{1\text{–}6}(x, y, z)$ are given in Appendix C.

Similarly, the solution for K_{II} has the form

$$K_{\mathrm{II}}(y) = \frac{P}{4\pi^2(1-\nu)\sqrt{2}}\,\mathrm{Re}\left(\sum_{m,n=x,y,z} \alpha_n \alpha_m \frac{\partial H_m}{\partial n}\right) \tag{58}$$

where $H_x \equiv \sqrt{l_1}(g_{10} - g_{11})$, $H_y \equiv \sqrt{l_1}\,i(g_{10} + g_{11})$, $H_z \equiv \sqrt{l_1}\,g_9$ and calculations yield

$$K_{\mathrm{II}}(y) = \frac{P\sqrt{l_1}}{4\pi^2(1-\nu)\sqrt{2}}\,\mathrm{Re}\,\{\alpha_x[\alpha_x(p_{10} - p_{11}) + \alpha_y(p_{12} - p_{13}) + \alpha_z(p_{14} - p_{15})] + i\alpha_y[\alpha_x(p_{10} + p_{11}) + \alpha_y(p_{12} + p_{13}) + \alpha_z(p_{14} + p_{15})] + \alpha_z[\alpha_x p_7 + \alpha_y p_8 + \alpha_z p_9]\} \tag{59}$$

where the elementary functions $p_{7\text{–}15}(x, y, z)$ are given in Appendix C.

The solution for K_{III} is obtained from the one for K_{II} by omitting the multiplier $1/(1-\nu)$ and replacing Re by Im in (59). Figure 6 illustrates these results.

Coplanar case. If the point of application of the dipole lies in the plane of the crack, the results simplify considerably (although calculations require finding non-trivial limits of the type 0/0).

Two cases should be distinguished: if the point of application of the dipole lies *outside* of the crack face, then $l_1 = 0$, $l_2 = 2x$; if the dipole is applied *on one of the crack faces*, then $l_1 = -2x$ and $l_2 = 0$. In the first case (the second case can be analyzed similarly), the results are as follows;

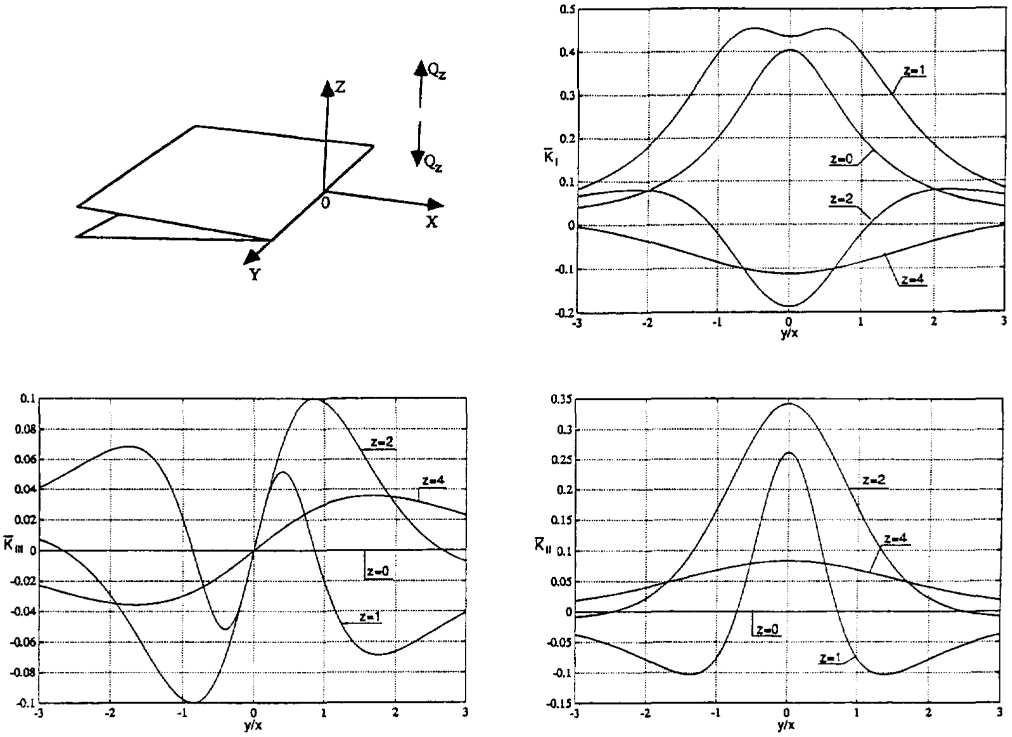

Fig. 6. Dimensionless stress intensity factors $\bar{K}_{(I,II,III)} = 4\pi^2\sqrt{2}x^{5/2}P^{-1}K_{(I,II,III)}$ along the crack edge due to a dipole oriented in the z-direction and applied at the point $(x = 1; y = 0; z)$ for several values of z. Poisson's ratio $\nu = 0.3$.

$$K_I(y) = \frac{P(1-2\nu)}{8\pi^2(1-\nu)}\frac{1}{R_0^2\sqrt{x}}\left\{\alpha_z^2+\alpha_x^2-\alpha_x\alpha_y\left[\frac{2xy}{R_0^2}-\frac{3}{R_0^2}\operatorname{Im}\left(t^2\left(\frac{2x}{\bar{t}}\right)^{1/2}\arctan\left(\frac{\bar{t}}{2x}\right)^{1/2}\right)\right]\right.$$
$$\left.-\frac{1}{2}(\alpha_x^2-\alpha_y^2)\left[\frac{2x^2}{R_0^2}-\frac{3}{R_0^2}\operatorname{Re}\left(t^2\left(\frac{2x}{\bar{t}}\right)^{1/2}\arctan\left(\frac{\bar{t}}{2x}\right)^{1/2}\right)\right]\right\} \quad (60)$$

$$K_{II}(y) = \frac{P}{4\pi^2(1-\nu)(2-\nu)}\frac{1}{R_0^2\sqrt{x}}\alpha_z\left\{\alpha_x\left[-2-\nu+\frac{4\nu y^2}{R_0^2}\right]-\alpha_y\frac{4\nu xy}{R_0^2}\right\} \quad (61)$$

$$K_{III}(y) = \frac{P}{4\pi^2(2-\nu)}\frac{1}{R_0^2\sqrt{x}}\alpha_z\left\{-\alpha_x\frac{4\nu xy}{R_0^2}+\alpha_y\left[3\nu-2-\frac{4\nu y^2}{R_0^2}\right]\right\} \quad (62)$$

where $R_0^2 = x^2+y^2$.

In particular, for a dipole in the z-direction,

$$K_I(y_0) = \frac{P(1-2\nu)}{8\pi^2(1-\nu)}\frac{1}{R_0^2\sqrt{x}}, \quad K_{II} = K_{III} = 0. \quad (63)$$

Note a similarity between (63) and the result for a half-plane crack interacting with a coplanar infinitesimal dislocation loop with Burgers' vector **b** in the z-direction (Rice (1985b)):

$$K_I = \frac{bE}{8\pi^2(1-\nu^2)}\frac{1}{R_0^2\sqrt{x}}. \quad (64)$$

(Rice's result differs from (64) by a multiplier $1/\sqrt{2\pi}$, due to a different definition of SIF.)

Comparison of (63) and (64) provides the equivalence relation between the dipole intensity and the magnitude of Burgers' vector: $P = bE/[(1+\nu)(1-2\nu)]$.

6. INTERACTION OF A HALF-PLANE CRACK WITH A CENTER OF DILATATION

Three mutually orthogonal dipoles of equal intensity P applied at the same point constitute a center of dilatation of intensity P. Such a stress source may be used to model a lattice vacancy, an interstitial atom or a foreign particle of the spherical shape.

Note that the stress field (and, therefore, SIFs) induced by such a source is identical to the one generated by a pressurized spherical cavity of radius b in an elastic continuum outside of the cavity (Lamé problem). The equivalence is established by the following relation between the pressure σ on the cavity and the dipole intensity:

$$\sigma = \frac{1-2\nu}{2\pi(1-\nu)b^3}P. \tag{65}$$

We consider a center of dilatation of intensity P applied at the point x, y, z and formed by three dipoles in mutually orthogonal directions characterized by directional cosines $(\alpha_x, \alpha_y, \alpha_z)$, $(\beta_x, \beta_y, \beta_z)$ and $(\gamma_x, \gamma_y, \gamma_z)$. The mode I SIF induced along the crack edge has the form:

$$K_{\rm I}(y) = \frac{P}{4\pi^2(1-\nu)\sqrt{2}} \sum_{m,n=x,y,z} (\alpha_n\alpha_m + \beta_n\beta_m + \gamma_n\gamma_m)\frac{\partial F_m}{\partial n}. \tag{66}$$

Since $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ are mutually orthogonal unit vectors, the expression in parentheses constitutes the nm-component of the unit tensor, i.e. it equals Kronecker's delta δ_{mn}. Thus,

$$K_{\rm I}(y) = \frac{P}{4\pi^2(1-\nu)\sqrt{2}}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right). \tag{67}$$

This result is independent of the orientation of the triad $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ (so that the dipoles can be assumed, for example, to be aligned with the x, y, z directions). It is this invariance that justifies the concept of a center of dilatation.

Calculating (67) and doing similar analysis for $K_{\rm II}$, $K_{\rm III}$ yield the following results

$$\begin{Bmatrix} K_{\rm I}(y) \\ K_{\rm II}(y) \\ K_{\rm III}(y) \end{Bmatrix} = P\frac{\sqrt{l_1}}{4\pi^2(1-\nu)\sqrt{2}}\,\mathrm{Re}\begin{Bmatrix} p_{16} \\ p_9 + p_{17} - p_{18} \\ -(1-\nu)i(p_9 + p_{17} - p_{18}) \end{Bmatrix} \tag{68}$$

in terms of elementary functions $p_9(x, y, z)$ and $p_{16\text{-}18}(x, y, z)$ given in Appendix C.

These results are illustrated in Fig. 7.

Coplanar case. In the case when the center of dilatation lies in the plane of the crack $z = 0$ (and is outside of the crack),

$$K_{\rm I}(y) = \frac{P(1-2\nu)}{4\pi^2(1-\nu)}\frac{1}{R_0^2\sqrt{x}}, \quad K_{\rm II} = K_{\rm III} = 0. \tag{69}$$

Note that the result in (69) is twice the value of $K_{\rm I}$ due to a coplanar dipole in the z direction.

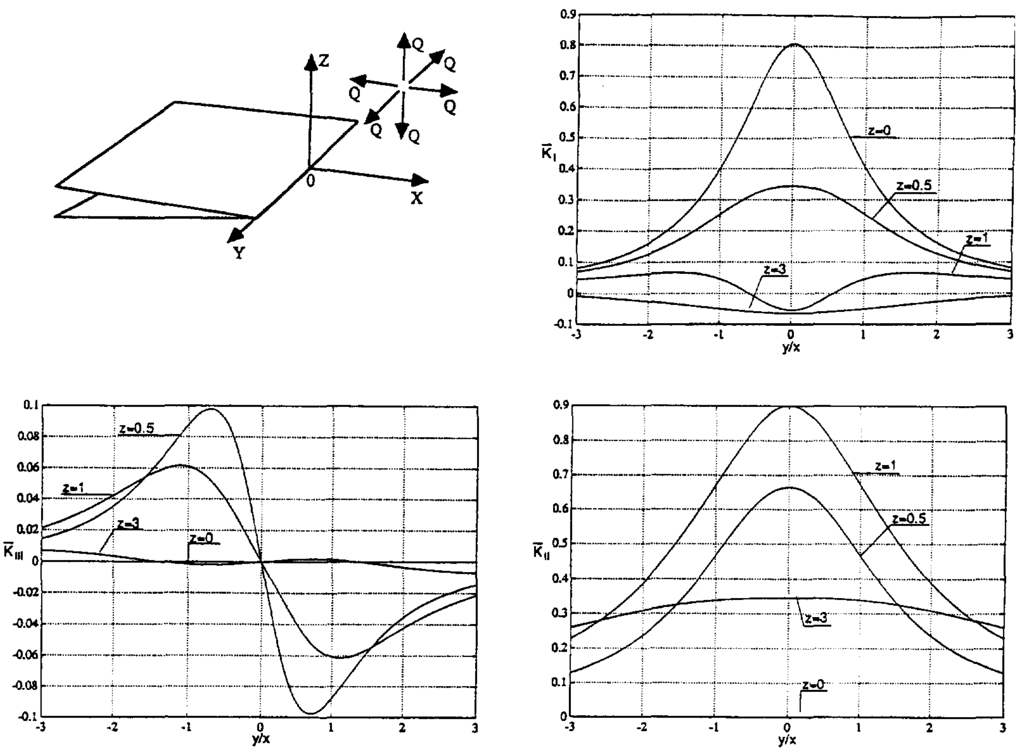

Fig. 7. Dimensionless stress intensity factors $\bar{K}_{(I,II,III)} = 4\pi^2\sqrt{2}x^{5/2}P^{-1}K_{(I,II,III)}$ along the crack edge due to a center of dilatation located at the point ($x = 1$; $y = 0$; z) for several values of z. Poisson's ration $\nu = 0.3$.

7. INTERACTION OF A HALF-PLANE CRACK WITH A MOMENT

A pair of equal and opposite point forces $\mathbf{Q}$ and $-\mathbf{Q}$, with the lines of action separated by the moment arm h and applied at the points where h intersects these lines, in the limit of $Q \to \infty$ and $h \to 0$, constitutes a concentrated moment. The limiting value of Qh (denoted by M) is the intensity of the moment.

The SIFs due to a moment can be obtained by differentiating the results for the point force in the direction normal to the force, i.e. along the moment arm h. Denoting the directional cosines of h by β_x, β_y, β_z and taking into account that $Q_k = Q\alpha_k$ ($k = x, y, z$), where α_k are directional cosines of the force direction, we obtain, for the mode I SIF:

$$K_I(y) = \frac{M}{4\pi^2(1-\nu)\sqrt{2}} \sum_{m,n=x,y,z} \beta_m \alpha_n \frac{\partial F_n}{\partial m} \tag{70}$$

where functions F_x, F_y, F_z are defined as in (56). Calculations yield the following result in terms of the elementary functions $p_{1-6}(x, y, z; y_0)$ given in Appendix C:

$$K_I(y) = \frac{M\sqrt{l_1}}{4\pi^2(1-\nu)\sqrt{2}} \{\alpha_x \operatorname{Re}[\beta_x p_1 + \beta_y p_2 + \beta_z p_3] + \alpha_y \operatorname{Im}[\beta_x p_1 + \beta_y p_2 + \beta_z p_3] + \alpha_z[\beta_x p_4 + \beta_y p_5 + \beta_z p_6]\}. \tag{71}$$

For K_{II}, the result can be obtained from (70) by replacing functions F_n by H_n (defined as in (58)) and taking the real part of the sum (in accordance with (41)):

$$K_{II}(y) = \frac{M\sqrt{l_1}}{4\pi^2(1-\nu)\sqrt{2}} \operatorname{Re}\{\alpha_x[\beta_x(p_{10}-p_{11}) + \beta_y(p_{12}-p_{13}) + \beta_z(p_{14}-p_{15})] + i\alpha_y[\beta_x(p_{10}+p_{11}) + \beta_y(p_{12}+p_{13}) + \beta_z(p_{14}+p_{15})] + \alpha_z[\beta_x p_7 + \beta_y p_8 + \beta_z p_9]\}. \tag{72}$$

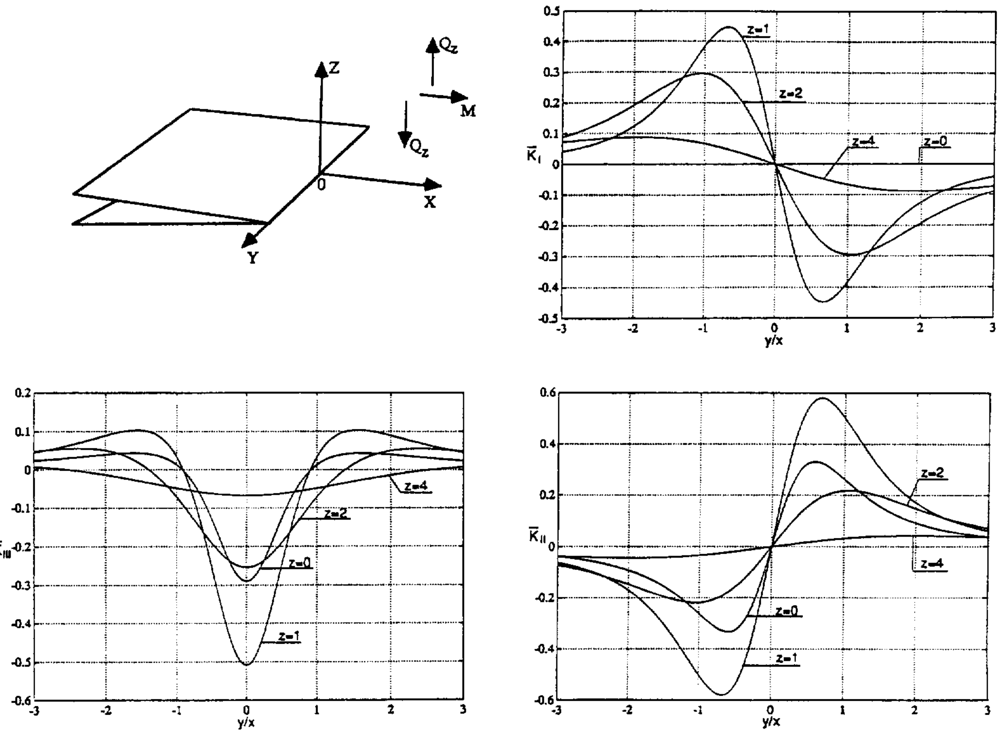

Fig. 8. Dimensionless stress intensity factors $\bar{K}_{(I,II,III)} = 4\pi^2\sqrt{2}x^{5/2}M^{-1}K_{(I,II,III)}$ along the crack edge due to a moment with the moment axis in the x-direction (produced by a pair of forces in the z-direction) applied at the point $(x = 1; y = 0; z)$ for several values of z. Poisson's ratio $\nu = 0.3$.

The solution for K_{III} is obtained from (72) by omitting the multiplier $1/(1-\nu)$ and replacing Re by Im.

Figure 8 illustrates the results.

Coplanar case. If the moment M is applied in the plane of the crack (outside of the crack), the results are as follows:

$$K_I(y) = \frac{M(1-2\nu)}{8\pi^2(1-\nu)}\frac{1}{R_0^2\sqrt{x}}\left\{-(\alpha_y\beta_x+\alpha_x\beta_y)\left[\frac{xy}{R_0^2}-\frac{3}{2R_0^2}\mathrm{Im}\left(t^2\left(\frac{2x}{\bar{t}}\right)^{1/2}\arctan\left(\frac{\bar{t}}{2x}\right)^{1/2}\right)\right]\right.$$
$$\left.+(\alpha_y\beta_y-\alpha_x\beta_x)\left[\frac{x^2}{R_0^2}-\frac{3}{2R_0^2}\mathrm{Re}\left(t^2\left(\frac{2x}{\bar{t}}\right)^{1/2}\arctan\left(\frac{\bar{t}}{2x}\right)^{1/2}\right)\right]+\alpha_z\beta_z+\alpha_x\beta_x\right\} \quad (73)$$

$$K_{II}(y) = \frac{M}{4\pi^2(1-\nu)(2-\nu)}\frac{1}{R_0^2\sqrt{x}}\left\{\beta_z\left[\alpha_x\left(-2-\nu+\frac{4\nu y^2}{R_0^2}\right)-\alpha_y\frac{4\nu xy}{R_0^2}\right]\right.$$
$$+(\alpha_z\beta_x-\alpha_x\beta_z)(1-2\nu)\left[\frac{3\nu}{2}+(1-3\nu)\frac{y^2}{R_0^2}+\frac{3(1-\nu)}{2R_0^2}\mathrm{Re}\left(t^2\left(\frac{2x}{\bar{t}}\right)^{1/2}\arctan\left(\frac{\bar{t}}{2x}\right)^{1/2}\right)\right]$$
$$\left.+(\alpha_y\beta_z-\alpha_z\beta_y)(1-2\nu)\left[(1-3\nu)\frac{xy}{R_0^2}-\frac{3(1-\nu)}{2R_0^2}\mathrm{Im}\left(t^2\left(\frac{2x}{\bar{t}}\right)^{1/2}\arctan\left(\frac{\bar{t}}{2x}\right)^{1/2}\right)\right]\right\}$$

(74)

$$K_{\rm III}(y) = \frac{M}{4\pi^2(2-\nu)}\frac{1}{R_0^2\sqrt{x}}\left\{\beta_z\left[-\alpha_x\frac{4\nu xy}{R_0^2}+\alpha_y\left(3\nu-2-\frac{4\nu y^2}{R_0^2}\right)\right]\right.$$

$$-(\alpha_z\beta_x-\alpha_x\beta_z)(1-2\nu)\left[\frac{xy}{R_0^2}-\frac{3}{2R_0^2}\mathrm{Im}\left(t^2\left(\frac{2x}{\bar{t}}\right)^{1/2}\arctan\left(\frac{\bar{t}}{2x}\right)^{1/2}\right)\right]$$

$$\left.-(\alpha_y\beta_z-\alpha_z\beta_y)(1-2\nu)\left[\frac{x^2}{R_0^2}-\frac{3}{2R_0^2}\mathrm{Re}\left(t^2\left(\frac{2x}{\bar{t}}\right)^{1/2}\arctan\left(\frac{\bar{t}}{2x}\right)^{1/2}\right)\right]\right\}. \quad (75)$$

8. INTERACTION OF A HALF-PLANE CRACK WITH A CENTER OF ROTATION

Center of rotation of intensity M is formed by two mutually orthogonal force pairs applied at the same point and producing moments of the same intensity M in the same direction.

The elastic field produced by such a stress source (and its impact on SIFs) is identical (outside of the sphere) to the one in the so-called Robin's problem, where a rigid sphere of radius b embedded into an elastic continuum is subjected to a rotation θ. This equivalence is established by the following relation between the rotation θ and the moment M:

$$\theta = \frac{M}{8\pi G b^3} \quad (76)$$

where G is the shear modulus.

Superimposing the result (70) and the one obtained from (70) by replacements $(\beta_x,\beta_y,\beta_z)\rightarrow(\alpha_x,\alpha_y,\alpha_z)$, $(\alpha_x,\alpha_y,\alpha_z)\rightarrow(-\beta_x,-\beta_y,-\beta_z)$ yields:

$$K_{\rm I}(y) = \frac{M}{4\pi^2(1-\nu)\sqrt{2}}\left\{(\alpha_y\beta_z-\alpha_z\beta_y)\left(\frac{\partial F_y}{\partial z}-\frac{\partial F_z}{\partial y}\right)+(\alpha_z\beta_x-\alpha_x\beta_z)\left(\frac{\partial F_z}{\partial x}-\frac{\partial F_x}{\partial z}\right)\right.$$

$$\left.+(\alpha_x\beta_y-\alpha_y\beta_x)\left(\frac{\partial F_x}{\partial y}-\frac{\partial F_y}{\partial x}\right)\right\} \quad (77)$$

or, since the expressions $\alpha_n\beta_m-\alpha_m\beta_n$ in (77) constitute the x, y and z components of the unit vector of the moment direction,

$$K_{\rm I}(y) = \frac{1}{4\pi^2(1-\nu)\sqrt{2}}\left\{M_x\left(\frac{\partial F_y}{\partial z}-\frac{\partial F_z}{\partial y}\right)+M_y\left(\frac{\partial F_z}{\partial x}-\frac{\partial F_x}{\partial z}\right)+M_z\left(\frac{\partial F_x}{\partial y}-\frac{\partial F_y}{\partial x}\right)\right\}. \quad (78)$$

An important observation is that $K_{\rm I}$ is expressed solely in terms of vector **M** and does not depend on the exact orientation (in the plane normal to **M**) of the two force pairs that constitute **M**. It is this invariance that justifies the concept of a center of rotation.

$K_{\rm I}$ can be further expressed in terms of the elementary functions $p_{1\text{–}5}$:

$$K_{\rm I}(y) = \frac{\sqrt{l_1}}{4\pi^2(1-\nu)\sqrt{2}}\{M_x(\mathrm{Im}\,p_3-p_5)+M_y(p_4-\mathrm{Re}\,p_3)+M_z(\mathrm{Re}\,p_2-\mathrm{Im}\,p_1)\}. \quad (79)$$

Similarly, for $K_{\rm II}$ we have:

$$K_{\rm II}(y) = \frac{\sqrt{l_1}}{4\pi^2(1-\nu)\sqrt{2}}\mathrm{Re}\,\{M_x[i(p_{14}+p_{15})-p_8]-M_y[p_{14}-p_{15}-p_7]$$

$$-M_z[i(p_{10}+p_{11})-p_{12}+p_{13}]\}. \quad (80)$$

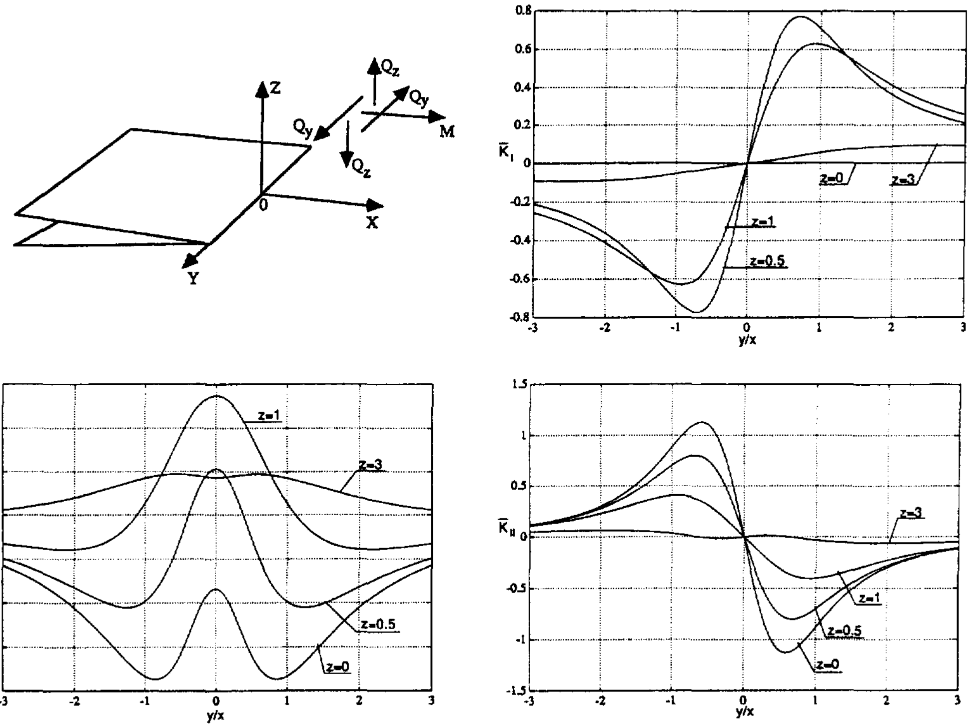

Fig. 9. Dimensionless stress intensity factors $\bar{K}_{(\mathrm{I,II,III})} = 4\pi^2\sqrt{2}x^{5/2}M^{-1}K_{(\mathrm{I,II,III})}$ along the crack edge due to a center of rotation with the axis of rotation in the x-direction (produced by two pairs of forces, in the z and y-directions) applied at the point $(x = 1\,; y = 0\,; z)$ for several values of z. Poisson's ratio $\nu = 0.3$.

The solution for K_{III} is obtained from the one for K_{II} by omitting the multiplier $1/(1-\nu)$ and replacing Re by Im in (80).

Figure 9 illustrates the results.

Coplanar case. If the point of application of the center of rotation lies in the plane of the crack $z = 0$ (outside of the crack),

$$K_{\mathrm{I}} = 0 \tag{81}$$

$$K_{\mathrm{II}}(y) = \frac{1}{4\pi^2(1-\nu)(2-\nu)}\frac{1}{R_0^2\sqrt{x}}\left\{2(1-\nu)(1+3\nu)M_y + \frac{2y}{R_0^2}[(1-2\nu)(1-3\nu)\right.$$
$$-2\nu](xM_x+yM_y) + \frac{3(1-\nu)(1-2\nu)}{R_0^2}\left[M_y \operatorname{Re}\left(t^2\left(\frac{2x}{\bar{t}}\right)^{1/2}\arctan\left(\frac{\bar{t}}{2x}\right)^{1/2}\right)\right.$$
$$\left.\left.-M_x \operatorname{Im}\left(t^2\left(\frac{2x}{\bar{t}}\right)^{1/2}\arctan\left(\frac{\bar{t}}{2x}\right)^{1/2}\right)\right]\right\} \tag{82}$$

$$K_{\mathrm{III}}(y) = \frac{1}{4\pi^2(2-\nu)}\frac{1}{R_0^2\sqrt{x}}\left\{-(4-7\nu)M_x + \frac{2y}{R_0^2}(1-4\nu)(yM_x - xM_y)\right.$$
$$+\frac{3(1-2\nu)}{R_0^2}\left[M_x \operatorname{Re}\left(t^2\left(\frac{2x}{\bar{t}}\right)^{1/2}\arctan\left(\frac{\bar{t}}{2x}\right)^{1/2}\right)\right. \tag{83}$$
$$\left.\left.+M_y \operatorname{Im}\left(t^2\left(\frac{2x}{\bar{t}}\right)^{1/2}\arctan\left(\frac{\bar{t}}{2x}\right)^{1/2}\right)\right]\right\}.$$

9. DISCUSSION

Interaction of a crack with several stress sources (or a continuous distribution of them) can, in principle, be analyzed by a summation (or integration) of the results for SIFs over all the sources. However, this may lead to either very lengthy expressions or to integrals that cannot be expressed in any standard functions.

If the system of forces is distributed over a volume V, which is *small*, as compared to its distance from the crack, then, to within values of higher order, the results can be obtained in a much simpler way: the impact of the force system can be reduced to the ones of the resultant vector, resultant moment and three mutually orthogonal dipoles, and the results of the present work can be utilized. Such a reduction is outlined, for example, by Lur'e (1964); see, also, Karapetian and Kachanov (1996).

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APPENDIX A

As mentioned in the Introduction, it is simpler to obtain the solution for the half-plane crack independently, rather than to apply a limiting procedure to the solutions for a circular crack as the crack radius $a \to \infty$. Nevertheless, the link between these two configurations can, in principle, be established, although it involves exceedingly difficult calculations of indeterminate ratios. Such a linkage is illustrated here on one of the simplest examples: the derivation of (45) for the mode I SIFs for a half-plane crack from the results for a circular crack.

The mode I SIFs for a *circular* crack due to arbitrary point forces (Q_x, Q_y, Q_z) applied at a point (ρ, ϕ, z) can be represented in the form (Karapetian and Kachanov (1996)):

$$K_{\mathrm{I}}(\phi_0) = \begin{Bmatrix} Q_x \\ Q_y \\ Q_z \end{Bmatrix} \frac{(a^2-l_1^2)^{1/2}}{4\pi^2(1-\nu)\sqrt{2a}} \begin{Bmatrix} \mathrm{Re} f_1 \\ \mathrm{Im} f_1 \\ f_2 \end{Bmatrix} \tag{A1}$$

where the elementary functions $f_{1,2}(\rho, \phi, z; \phi_0)$ are as follows:

$$f_1 = \frac{1}{\bar{q}}\left[\frac{z}{R^2}\left(2(1-\nu) - \frac{2z^2}{R^2} + \frac{\rho^2 - l_1^2}{l_2^2 - l_1^2}\right) + \frac{z l_2^2}{(l_2^2 - a^2 + \bar{s}^2)(l_2^2 - l_1^2)} - (1-2\nu)\frac{a}{\bar{s}(a^2 - l_1^2)^{1/2}} \arctan\left(\frac{\bar{s}}{(l_2^2 - a^2)^{1/2}}\right)\right] \tag{A2}$$

$$f_2 = \frac{1}{R^2}\left[2(1-\nu) + \frac{2z^2}{R^2} - \frac{\rho^2 - l_1^2}{l_2^2 - l_1^2}\right] \tag{A3}$$

and the following notations are used:

$$R^2 = \rho^2 + a^2 - 2\rho a \cos(\phi - \phi_0) + z^2 = q\bar{q} + z^2, \quad s^2 = a^2 - a\rho\, e^{i(\phi - \phi_0)}, \quad q = \rho\, e^{i\phi} - a\, e^{i\phi_0}$$

$$2l_1 = \sqrt{(a+\rho)^2 + z^2} - \sqrt{(a-\rho)^2 + z^2}, \quad 2l_2 = \sqrt{(a+\rho)^2 + z^2} + \sqrt{(a-\rho)^2 + z^2}. \tag{A4}$$

We use the following results on calculation of four limits (Fabrikant *et al.* (1993)):

$$\lim_{a\to\infty}\left(\frac{a^2 - l_1^2}{a}\right) = \sqrt{x^2 + z^2} - x \equiv l_1, \quad \lim_{a\to\infty}\left(\frac{l_2^2 - a^2}{a}\right) = \sqrt{x^2 + z^2} + x \equiv l_2$$

$$\lim_{a\to\infty}\left(\frac{s^2}{a}\right) = -x - i(y - y_0) \equiv t, \quad \lim_{a\to\infty}\left(\frac{a^2 - \rho^2}{a}\right) = -2x \tag{A5}$$

where y_0 is the coordinate of a point of the edge of the half-plane crack which corresponds to the point ϕ_0 of the edge of the circular crack. Without loss of generality, we can assume $y_0 = 0$.

Since $\rho\, e^{i\varphi} = x + a + iy$ and $a\, e^{i\varphi_0} = a + iy_0 = a$, the values of R and q, as given by (A4), take the form:

$$R = \sqrt{x^2 + y^2 + z^2}, \quad q = x + iy \equiv -t. \tag{A6}$$

We now rearrange (A2) as follows:

$$f_1 = \frac{1}{\bar{q}}\left[\frac{z}{R^2}\left(2(1-\nu) - \frac{2z^2}{R^2} + \frac{(\rho^2 - a^2)/a + (a^2 - l_1^2)/a}{(l_2^2 - a^2)/a + (a^2 - l_1^2)/a}\right) + \frac{z l_2^2/a^2}{((l_2^2 - a^2)/a + \bar{s}^2/a)((l_2^2 - a^2)/a + (a^2 - l_1^2)/a)}\right.$$
$$\left. - (1-2\nu)\frac{1}{(\bar{s}/\sqrt{a})(\sqrt{a^2 - l_1^2}/\sqrt{a})} \arctan\left(\frac{\bar{s}/\sqrt{a}}{\sqrt{l_2^2 - a^2}/\sqrt{a}}\right)\right]$$

so that $\lim_{a\to\infty} f_1$ yields the function g_8 given by (49). Similarly, it can be shown that $\lim_{a\to\infty} f_2$ yields g_7 given by (48). Thus, the solution for a *half-plane* crack is recovered.

APPENDIX B

Derivation of the formulas (32) and (33) is given here. It follows the same general logic as the one in the work of Karapetian and Hanson (1994). We first rewrite (6) in the form:

$$u_z = \mathrm{Re}(AT) = (1/2)(AT + \overline{AT}), \tag{B1}$$

where A is a complex-valued coefficient. The displacement u_z at the point (x, y, z) due to a pair of equal and opposite point forces T_x applied at the points $(x_0, 0, \pm 0)$ of the crack faces is:

$$u_z^{T_x} = T_x(1/2)(A + \bar{A}) = T_x\, \mathrm{Re}(A). \tag{B2}$$

The displacement u_z due to a similar pair of forces T_y is:

$$u_z^{T_y} = iT_y(1/2)(A - \bar{A}) = -T_y\, \mathrm{Im}(A). \tag{B3}$$

Denoting the tangential displacement discontinuities in the x and y directions due to a point force Q_z applied at the point (x, y, z) by $[u_x^{Q_z}]$ and $[u_y^{Q_z}]$, we obtain, from the reciprocity theorem,

$$[u_x^{Q_z}] = Q_z \operatorname{Re}(A), \quad [u_y^{Q_z}] = -Q_z \operatorname{Im}(A) \tag{B4}$$

so that

$$\Delta^{Q_z} \equiv [u_x^{Q_z}] + i[u_y^{Q_z}] = Q_z[Re(A) - i \operatorname{Im}(A)] = \bar{A}Q_z. \tag{B5}$$

A comparison of equations (B1) and (B5) shows that the tangential displacement discontinuity at the point $(x_0, 0, \pm 0)$ due to Q_z applied at the point (x, y, z) can be expressed in the form (32).

Derivation of (33) is done in a similar manner. We rewrite (7) in the form:

$$u_x + iu_y = B_1 T + B_2 \bar{T} \tag{B6}$$

where B_1 and B_2 are the complex-valued coefficients. The displacements u_x, u_y at the point (x, y, z) due to a pair of equal and opposite point forces T_x applied at $(x_0, 0, \pm 0)$ are:

$$u_x^{T_x} = T_x \operatorname{Re}(B_1 + B_2), \quad u_y^{T_x} = T_x \operatorname{Im}(B_1 + B_2), \tag{B7}$$

and the displacements due to a pair of forces T_y are:

$$u_x^{T_y} = T_y \operatorname{Re}[(B_1 - B_2)i] = -T_y \operatorname{Im}(B_1 - B_2), \quad u_y^{T_y} = T_y \operatorname{Im}[(B_1 - B_2)i] = T_y \operatorname{Re}(B_1 - B_2). \tag{B8}$$

Denoting the tangential displacement discontinuity in the x and y directions due to force Q_x applied at the point (x, y, z) by $[u_x^{Q_x}]$ and $[u_y^{Q_x}]$, we obtain, from the reciprocity theorem:

$$[u_x^{Q_x}] = Q_x \operatorname{Re}(B_1 + B_2), \quad [u_y^{Q_x}] = -Q_x \operatorname{Im}(B_1 - B_2) \tag{B9}$$

or

$$\Delta^{Q_x} \equiv [u_x^{Q_x}] + i[u_y^{Q_x}] = Q_x(\bar{B}_1 + B_2). \tag{B10}$$

Displacement discontinuities due to Q_y are obtained in the same way:

$$[u_x^{Q_y}] = Q_y \operatorname{Im}(B_1 + B_2), \quad [u_y^{Q_y}] = Q_y \operatorname{Re}(B_1 - B_2) \tag{B11}$$

or

$$\Delta^{Q_y} \equiv [u_x^{Q_y}] + i[u_y^{Q_y}] = iQ_y(\bar{B}_1 - B_2). \tag{B12}$$

Hence,

$$\Delta^{Q_x} + \Delta^{Q_y} = \bar{B}_1 Q + B_2 \bar{Q}. \tag{B13}$$

A comparison of (B6) and (B13) shows that the tangential displacement discontinuity at the point $(x_0, 0, 0^{\pm})$ due to the point force (Q_x, Q_y) parallel to the crack and applied at (x, y, z) can be expressed in the form (33).

APPENDIX C

Expressions for the functions $p_i = p_i(x, y, z)$ entering the solutions in the main text are given here.

$$p_1 = \frac{1}{\sqrt{l_1}} \frac{\partial}{\partial x}(\sqrt{l_1} g_8) = \frac{z}{\bar{t}R^2}\left[\left(\frac{2x}{R^2} + \frac{1}{l_1 + l_2} - \frac{1}{\bar{t}}\right)\left(2(1-\nu) - \frac{2z^2}{R^2} + \frac{l_2}{l_1 + l_2}\right) - \frac{4xz^2}{R^4} - \frac{4z^2}{(l_1 + l_2)^3}\right]$$
$$+ \frac{z}{\bar{t}(l_1 + l_2)^2(l_2 + \bar{t})}\left[1 - (1-2\nu)\frac{l_1 + l_2}{l_1} - \frac{l_1 + l_2}{\bar{t}} + \frac{4x}{l_1 + l_2} + \frac{2x}{l_2 + \bar{t}}\right]$$
$$+ \frac{1-2\nu}{2\bar{t}^2 z}\left[3\left(\frac{l_2}{\bar{t}}\right)^{1/2} \arctan\left(\frac{\bar{t}}{l_2}\right)^{1/2} - \frac{l_2}{l_2 + \bar{t}}\right]$$

$$p_2 = \frac{1}{\sqrt{l_1}} \frac{\partial}{\partial y}(\sqrt{l_1} g_8) = i\left\{\frac{z}{\bar{t}R^2}\left[\left(\frac{1}{\bar{t}} - \frac{2iy}{R^2}\right)\left(2(1-\nu) - \frac{2z^2}{R^2} + \frac{l_2}{l_1 + l_2}\right) + \frac{4iyz^2}{R^4}\right]\right.$$
$$\left. + \frac{z}{\bar{t}(l_1 + l_2)(l_2 + \bar{t})}\left(\frac{1}{l_2 + \bar{t}} + \frac{1}{\bar{t}}\right) - \frac{1-2\nu}{2\bar{t}^2 z}\left[3\left(\frac{l_2}{\bar{t}}\right)^{1/2} \arctan\left(\frac{\bar{t}}{l_2}\right)^{1/2} - \frac{l_2}{l_2 + \bar{t}}\right]\right\}$$

$$p_3 = \frac{1}{\sqrt{l_1}}\frac{\partial}{\partial z}(\sqrt{l_1}g_8) = \frac{1}{\bar{t}R^2}\left[\frac{4z^2}{R^2}\left(1-\frac{z^2}{R^2}\right)-\left(1-\frac{2z^2}{R^2}+\frac{l_2}{l_1+l_2}\right)\left(2(1-v)-\frac{2z^2}{R^2}+\frac{l_2}{l_1+l_2}\right)\right.$$
$$\left.+\frac{4z^2x}{(l_1+l_2)^3}\right]-\frac{1}{\bar{t}(l_1+l_2)(l_2+\bar{t})}\left[2(1-v)+\frac{l_2}{l_1+l_2}-\frac{4z^2}{(l_1+l_2)^2}-\frac{2z^2}{(l_1+l_2)(l_2+\bar{t})}\right]$$

$$p_4 = \frac{1}{\sqrt{l_1}}\frac{\partial}{\partial x}(\sqrt{l_1}g_7) = \frac{1}{R^2}\left[\frac{2z^2}{l_1+l_2}\left(\frac{1}{R^2}-\frac{2}{(l_1+l_2)^2}\right)-\left(\frac{2x}{R^2}-\frac{1}{l_1+l_2}\right)\left(2(1-v)+\frac{4z^2}{R^2}-\frac{l_2}{l_1+l_2}\right)\right]$$

$$p_5 = \frac{1}{\sqrt{l_1}}\frac{\partial}{\partial y}(\sqrt{l_1}g_7) = -\frac{2y}{R^4}\left[2(1-v)+\frac{4z^2}{R^2}-\frac{l_2}{l_1+l_2}\right]$$

$$p_6 = \frac{1}{\sqrt{l_1}}\frac{\partial}{\partial z}(\sqrt{l_1}g_7) = \frac{1}{R^2z}\left[\left(2(1-v)+\frac{2z^2}{R^2}-\frac{l_2}{l_1+l_2}\right)\left(\frac{l_2}{l_1+l_2}-\frac{2z^2}{R^2}\right)+\frac{4z^2(R^2-z^2)}{R^4}+\frac{4z^2x}{(l_1+l_2)^3}\right]$$

$$p_7 = \frac{1}{\sqrt{l_1}}\frac{\partial}{\partial x}(\sqrt{l_1}g_9) = \frac{z}{R^2\bar{t}}\left[\left(2v-\frac{2z^2}{R^2}+\frac{l_2}{l_1+l_2}\right)\left(\frac{1}{\bar{t}}-\frac{2x}{R^2}-\frac{1}{l_1+l_2}\right)+\frac{4z^2x}{R^4}+\frac{4z^2}{(l_1+l_2)^3}\right]$$
$$-\frac{v(1-2v)}{2-v}\left[\frac{3}{2t^2\sqrt{tl_1}}\arctan\left(\frac{t}{l_2}\right)^{1/2}-\frac{z}{tl_1(l_2+t)^2}\left(1+\frac{2t}{l_1+l_2}+\frac{3(l_2+t)}{2t}\right)\right]$$
$$+\frac{v}{2-v}\frac{2z}{(l_1+l_2)^2(l_2+t)^2}\left(1+\frac{4x}{l_1+l_2}+\frac{4x}{l_2+t}\right)+(1-2v)\frac{3}{2\bar{t}^2\sqrt{\bar{t}l_1}}\arctan\left(\frac{\bar{t}}{l_2}\right)^{1/2}$$
$$-\frac{z}{\bar{t}l_1(l_1+l_2)(l_2+\bar{t})}\left[(1-2v)\left(1+\frac{l_1+l_2}{2\bar{t}}\right)+\frac{l_1}{l_1+l_2}-\frac{l_1}{\bar{t}}+\frac{4xl_1}{(l_1+l_2)^2}+\frac{2xl_1}{(l_1+l_2)(l_2+\bar{t})}\right]$$

$$p_8 = \frac{1}{\sqrt{l_1}}\frac{\partial}{\partial y}(\sqrt{l_1}g_9) = i\left\{\frac{z}{R^2\bar{t}}\left[\left(2v-\frac{2z^2}{R^2}+\frac{l_2}{l_1+l_2}\right)\left(\frac{2iy}{R^2}-\frac{1}{\bar{t}}\right)-\frac{4iyz^2}{R^4}\right]\right.$$
$$-\frac{v(1-2v)}{2-v}\left[\frac{3}{2t^2\sqrt{tl_1}}\arctan\left(\frac{t}{l_2}\right)^{1/2}-\frac{z}{tl_1(l_2+t)^2}\left(1+\frac{3(l_2+t)}{2t}\right)\right]-\frac{v}{2-v}\frac{4z}{(l_1+l_2)(l_2+t)^3}$$
$$\left.-(1-2v)\frac{3}{2\bar{t}^2\sqrt{\bar{t}l_1}}\arctan\left(\frac{\bar{t}}{l_2}\right)^{1/2}+\frac{z}{\bar{t}l_1(l_1+l_2)(l_2+\bar{t})}\left[(1-2v)\frac{l_1+l_2}{2\bar{t}}-\frac{l_1}{\bar{t}}-\frac{l_1}{l_2+\bar{t}}\right]\right\}$$

$$p_9 = \frac{1}{\sqrt{l_1}}\frac{\partial}{\partial z}(\sqrt{l_1}g_9) = \frac{1}{R^2\bar{t}}\left[\left(2v-\frac{2z^2}{R^2}+\frac{l_2}{l_1+l_2}\right)\left(1-\frac{2z^2}{R^2}+\frac{l_2}{l_1+l_2}\right)-\frac{4z^2}{R^2}\left(1-\frac{z^2}{R^2}\right)-\frac{4z^2x}{(l_1+l_2)^3}\right]$$
$$+\frac{1}{\bar{t}(l_1+l_2)(l_2+\bar{t})}\left[2v+\frac{l_2}{l_1+l_2}-\frac{4z^2}{(l_1+l_2)^2}-\frac{2z^2}{(l_1+l_2)(l_2+\bar{t})}\right]$$
$$-\frac{2v}{2-v}\frac{1}{(l_1+l_2)(l_2+t)^2}\left[2v+\frac{l_2}{l_1+l_2}-\frac{4z^2}{(l_1+l_2)^2}-\frac{4z^2}{(l_1+l_2)(l_2+t)}\right]$$

$$p_{10} = \frac{1}{\sqrt{l_1}}\frac{\partial}{\partial x}(\sqrt{l_1}g_{10}) = \frac{1}{2R^2}\left[\left(2(2-v)-\frac{2z^2}{R^2}+\frac{l_2}{l_1+l_2}\right)\left(\frac{2x}{R^2}+\frac{1}{l_1+l_2}\right)-\frac{4z^2x}{R^4}-\frac{4z^2}{(l_1+l_2)^3}\right]$$
$$+\frac{v}{2-v}\frac{1}{2t}\left\{(1-2v)\frac{3}{2t^2}\left[5\left(\frac{l_2}{t}\right)^{1/2}\arctan\left(\frac{t}{l_2}\right)^{1/2}-\frac{l_2}{l_2+t}\left(1+\frac{2t}{l_1+l_2}\right)\right]\right.$$
$$+\frac{3z^2}{(l_1+l_2)^2(l_2+t)^2}\left(1-\frac{l_1+l_2}{t}+\frac{4x}{l_1+l_2}+\frac{4x}{l_2+t}\right)+\left[2v-\frac{l_2}{l_1+l_2}-\frac{z^2}{(l_1+l_2)(l_2+t)}\right]\left[\frac{6}{t^2}\right.$$
$$\left.-\frac{1}{t(l_2+t)}-\frac{3}{t(l_1+l_2)}+\frac{1}{(l_1+l_2)(l_2+t)}+\frac{2x}{(l_1+l_2)(l_2+t)^2}\right]$$
$$\left.-\frac{2z^2}{(l_1+l_2)^2}\left(\frac{3}{t}-\frac{1}{l_2+t}\right)\left[\frac{2}{l_1+l_2}-\frac{2x}{(l_1+l_2)(l_2+t)}-\frac{x}{(l_2+t)^2}\right]\right\}$$

$$p_{11} = \frac{1}{\sqrt{l_1}}\frac{\partial}{\partial x}(\sqrt{l_1}g_{11}) = \frac{1}{2\bar t^2}\Big\{\frac{2z^2x}{R^4}\Big(2(1-\nu)-\frac{4z^2}{R^2}+\frac{l_2}{l_1+l_2}\Big)-\frac{4z^2}{(l_1+l_2)^3}\Big(1+\frac{z^2}{R^2}\Big)$$

$$+(1-2\nu)\frac{3}{2\bar t}\Big[5\Big(\frac{l_2}{\bar t}\Big)^{1/2}\arctan\Big(\frac{\bar t}{l_2}\Big)^{1/2}-\frac{l_2}{l_2+\bar t}\Big(1+\frac{2\bar t}{l_1+l_2}\Big)\Big]-\frac{l_2}{l_1+l_2}\Big(1+\frac{2}{2-\nu}\frac{\bar t}{l_1-\bar t}$$

$$+\frac{\nu}{2-\nu}\frac{\bar t}{l_2+t}\Big)\Big[\frac{2}{\bar t}-\frac{1}{l_1+l_2}+\frac{4l_1}{(l_1+l_2)^2}\Big]+\frac{6z^2}{(l_1+l_2)^2(l_2+\bar t)}\Big(\frac{1}{2}-\frac{l_1+l_2}{\bar t}+\frac{x}{l_2+\bar t}+\frac{2x}{l_1+l_2}\Big)$$

$$+\frac{\nu}{2-\nu}\frac{z^2}{(l_1+l_2)(l_2+t)}\Big(\frac{1}{l_1}+\frac{2}{l_2+t}\Big)+\frac{2}{2-\nu}\frac{l_2}{(l_1+l_2)(l_1-\bar t)}\Big[1+\frac{2\bar t x}{(l_1+l_2)(l_1-\bar t)}\Big]$$

$$+\Big(\frac{2}{\bar t}-\frac{1}{l_1+l_2}\Big)\Big[\Big(1+\frac{z^2}{R^2}\Big)\Big(2\nu-\frac{l_2}{l_1+l_2}\Big)+2\nu-\frac{2z^2}{R^2}\Big(1-\frac{z^2}{R^2}\Big)\Big]+\frac{\bar t}{(l_1+l_2)(l_2+t)}\Big[2\nu\Big(1-\frac{l_1+l_2}{\bar t}$$

$$+\frac{2x}{l_2+t}\Big)-\frac{\nu}{2-\nu}\frac{2z^2}{(l_1+l_2)(l_2+t)}\Big(1-\frac{x}{l_1}+\frac{4x}{l_2+t}+\frac{4x}{l_1+l_2}\Big)\Big]\Big\}$$

$$p_{12} = \frac{1}{\sqrt{l_1}}\frac{\partial}{\partial y}(\sqrt{l_1}g_{10}) = \frac{y}{R^4}\Big(2(2-\nu)-\frac{4z^2}{R^2}+\frac{l_2}{l_1+l_2}\Big)$$

$$+\frac{\nu}{2-\nu}\frac{i}{2t^2}\Big[(1-2\nu)\frac{3}{2t}\Big(5\Big(\frac{l_2}{t}\Big)^{1/2}\arctan\Big(\frac{t}{l_2}\Big)^{1/2}-\frac{l_2}{l_2+t}\Big)-\frac{z^2t}{(l_1+l_2)(l_2+t)^2}\Big(\frac{6}{t}+\frac{5}{l_2+t}\Big)$$

$$+\Big(\frac{6}{t}-\frac{1}{l_2+t}-\frac{t}{(l_2+t)^2}\Big)\Big(2\nu-\frac{l_2}{l_1+l_2}-\frac{z^2}{(l_1+l_2)(l_2+t)}\Big)\Big]$$

$$p_{13} = \frac{1}{\sqrt{l_1}}\frac{\partial}{\partial y}(\sqrt{l_1}g_{11}) = \frac{i}{2\bar t^2}\Big\{\frac{2z^2}{R^2}\Big(\frac{1}{\bar t}-\frac{iy}{R^2}\Big)\Big(2(1-\nu)-\frac{4z^2}{R^2}+\frac{l_2}{l_1+l_2}\Big)$$

$$-\frac{4}{\bar t}\Big(2\nu-\frac{z^4}{R^4}-\frac{l_2}{l_1+l_2}\Big)-(1-2\nu)\frac{3}{2\bar t}\Big[5\Big(\frac{l_2}{\bar t}\Big)^{1/2}\arctan\Big(\frac{\bar t}{l_2}\Big)^{1/2}-\frac{l_2}{l_2+\bar t}\Big]$$

$$+\frac{3z^2}{(l_1+l_2)(l_2+\bar t)}\Big(\frac{2}{\bar t}+\frac{1}{l_2+\bar t}\Big)+2\nu\frac{1}{l_2+t}\Big(2-\frac{l_1}{l_2+t}\Big)+\frac{2}{2-\nu}\frac{z^2}{(l_1+l_2)(l_1-\bar t)}\Big(\frac{2}{l_1}-\frac{1}{l_1-\bar t}\Big)$$

$$+\frac{\nu}{2-\nu}\frac{z^2}{(l_1+l_2)(l_2+t)^2}\Big[\frac{2\bar t}{l_2+t}-5+\frac{2(l_2+t)}{l_1}+\frac{2l_1}{l_2+t}\Big]\Big\}$$

$$p_{14} = \frac{1}{\sqrt{l_1}}\frac{\partial}{\partial z}(\sqrt{l_1}g_{10}) = \frac{1}{R^2z}\Big[\Big(2(2-\nu)-\frac{2z^2}{R^2}+\frac{l_2}{l_1+l_2}\Big)\Big(\frac{z^2}{R^2}-\frac{l_2}{2(l_1+l_2)}\Big)+\frac{2z^2}{R^2}\Big(1-\frac{z^2}{R^2}\Big)$$

$$+\frac{2z^2x}{(l_1+l_2)^3}\Big]+\frac{\nu}{2-\nu}\frac{1}{2t^2z}\Big\{3(1-2\nu)\Big[\Big(\frac{l_2}{t}\Big)^{1/2}\arctan\Big(\frac{t}{l_2}\Big)^{1/2}-\frac{z^2}{(l_1+l_2)(l_2+t)}\Big]$$

$$+\frac{l_2}{l_1+l_2}\Big[2\nu-\frac{l_2}{l_1+l_2}-\frac{z^2}{(l_1+l_2)(l_2+t)}\Big]\Big[2+\frac{l_2}{l_2+t}+\frac{2l_1t}{(l_2+t)^2}\Big]+\frac{4z^2x}{(l_1+l_2)^3}\Big(2+\frac{l_2}{l_2+t}\Big)$$

$$-\frac{6z^2}{(l_1+l_2)(l_2+t)}\Big[1-\frac{2z^2}{(l_1+l_2)^2}-\frac{z^2}{(l_1+l_2)(l_2+t)}\Big]$$

$$-\frac{z^2t}{(l_1+l_2)(l_2+t)^2}\Big[4+\frac{3l_2}{l_1+l_2}-\frac{8z^2}{(l_1+l_2)^2}-\frac{10z^2}{(l_1+l_2)(l_2+t)}\Big]\Big\}$$

$$p_{15} = \frac{1}{\sqrt{l_1}}\frac{\partial}{\partial z}(\sqrt{l_1}g_{11}) = \frac{1}{2\bar t^2}\Big\{\frac{z}{R^2}\Big(2-\frac{2z^2}{R^2}+\frac{l_2}{l_1+l_2}\Big)\Big(\frac{4z^2}{R^2}-\frac{l_2}{l_1+l_2}-2(1-\nu)\Big)$$

$$+\frac{z}{(l_1+l_2)l_1}\Big(4\nu-\frac{2z^4}{R^4}-\frac{l_2}{l_1+l_2}\Big)+\frac{4zx}{(l_1+l_2)^3}\Big(2+\frac{z^2}{R^2}+\frac{2}{2-\nu}\frac{\bar t}{l_1-\bar t}\Big)$$

$$+3(1-2\nu)\left[\frac{1}{\sqrt{\bar{t}l_1}}\arctan\left(\frac{\bar{t}}{l_2}\right)^{1/2}-\frac{z}{(l_1+l_2)(l_2+\bar{t})}\right]-\frac{zl_2}{(l_1+l_2)^2l_1}\left(1-\frac{2}{2-\nu}\frac{\bar{t}(l_1+\bar{t})}{(l_1-\bar{t})^2}\right)$$

$$-\frac{3z}{(l_1+l_2)(l_2+\bar{t})}\left[2+\frac{l_2}{l_1+l_2}-\frac{4z^2}{(l_1+l_2)^2}-\frac{2z^2}{(l_1+l_2)(l_2+\bar{t})}\right]$$

$$+\frac{\nu}{2-\nu}\frac{4z\bar{t}}{(l_1+l_2)(l_2+t)}\left[\frac{x}{(l_1+l_2)^2}+\frac{1}{l_2+t}-\frac{2z^2}{(l_1+l_2)^2(l_2+t)}-\frac{z^2}{(l_1+l_2)(l_2+t)^2}\right]$$

$$+\frac{z\bar{t}(l_1+\bar{t})}{(l_1+l_2)(l_2+t)^2l_1}\left[2\nu+\frac{\nu}{2-\nu}\left(\frac{l_2}{l_1+l_2}-\frac{2z^2}{(l_1+l_2)(l_2+t)}\right)\right]\Bigg\}$$

$$p_{16}=\frac{1}{R^2z}\left[\frac{l_2}{l_1+l_2}\left(2(1-\nu)+\frac{2z^2}{R^2}-\frac{l_2}{l_1+l_2}+\frac{4l_1x}{(l_1+l_2)^2}\right)-(1-2\nu)\frac{4z^2}{R^2}\right]$$

$$+\frac{z}{\bar{t}(l_1+l_2)}\left[\frac{1}{R^2}\left(2(1-\nu)-\frac{2z^2}{R^2}+\frac{l_2}{l_1+l_2}-\frac{4z^2}{(l_1+l_2)^2}\right)\right.$$

$$\left.+\frac{1}{(l_1+l_2)(l_2+\bar{t})}\left(1-(1-2\nu)\frac{l_1+l_2}{l_1}+\frac{4x}{l_1+l_2}+\frac{2l_2}{l_2+\bar{t}}\right)\right]$$

$$p_{17}=\frac{1}{R^2}\left[\left(2(2-\nu)-\frac{2z^2}{R^2}+\frac{l_2}{l_1+l_2}\right)\left(\frac{1}{2(l_1+l_2)}-\frac{t}{R^2}\right)+\frac{2z^2t}{R^4}-\frac{2z^2}{(l_1+l_2)^3}\right]$$

$$-\frac{\nu}{2-\nu}\frac{1}{2t^2}\left\{(1-2\nu)\frac{3l_2}{(l_1+l_2)(l_2+t)}-\frac{3z^2t}{(l_1+l_2)^2(l_2+t)^2}\left[1+\frac{l_1+l_2}{t}+\frac{4x}{l_1+l_2}+\frac{4x}{l_2+t}\right.\right.$$

$$\left.+\frac{5(l_1+l_2)}{3(l_2+t)}\right]-\left(2\nu-\frac{l_2}{l_1+l_2}-\frac{z^2}{(l_1+l_2)(l_2+t)}\right)\left[\frac{t}{(l_2+t)^2}-\frac{3}{l_1+l_2}+\frac{t}{(l_1+l_2)(l_2+t)}\right.$$

$$\left.\left.+\frac{2xt}{(l_1+l_2)(l_2+t)^2}\right]+\frac{2z^2}{(l_1+l_2)^2}\left(2+\frac{l_2}{l_2+t}\right)\left[\frac{2}{l_1+l_2}-\frac{x}{(l_2+t)^2}-\frac{2x}{(l_1+l_2)(l_2+t)}\right]\right\}$$

$$p_{18}=\frac{1}{2\bar{t}^2}\left\{\frac{2z^2}{R^2}\left(\frac{1}{\bar{t}}-\frac{\bar{t}}{R^2}\right)\left(2(1-\nu)-\frac{4z^2}{R^2}+\frac{l_2}{l_1+l_2}\right)-\frac{4}{\bar{t}}\left(2\nu-\frac{z^4}{R^4}-\frac{l_2}{l_1+l_2}\right)\right.$$

$$-\frac{4z^2}{(l_1+l_2)^3}\left(1+\frac{z^2}{R^2}\right)+\left(\frac{2}{\bar{t}}-\frac{1}{l_1+l_2}\right)\left[\left(1+\frac{z^2}{R^2}\right)\left(2\nu-\frac{l_2}{l_1+l_2}\right)+2\nu-\frac{2z^2}{R^2}\left(1-\frac{z^2}{R^2}\right)\right]$$

$$+\frac{3z^2}{(l_1+l_2)^2(l_2+\bar{t})}\left[1-(1-2\nu)\frac{l_1+l_2}{l_1}+\frac{4x}{l_1+l_2}+\frac{2l_2}{l_2+\bar{t}}\right]+\frac{2}{2-\nu}\frac{l_2}{(l_1+l_2)(l_1-\bar{t})}\left[3-\frac{l_1}{l_1-\bar{t}}\right.$$

$$\left.+\frac{2\bar{t}x}{(l_1+l_2)(l_1-\bar{t})}\right]-\frac{\nu}{2-\nu}\frac{z^2}{(l_1+l_2)(l_2+t)^2}\left[3-\frac{3(l_2+t)}{l_1}-\frac{2(l_1+\bar{t})}{l_2+t}\right]-\frac{l_2}{l_1+l_2}\left(1+\frac{2}{2-\nu}\frac{\bar{t}}{l_1-\bar{t}}\right.$$

$$\left.+\frac{\nu}{2-\nu}\frac{\bar{t}}{l_2+t}\right)\left(\frac{2}{\bar{t}}-\frac{1}{l_1+l_2}+\frac{4l_1}{(l_1+l_2)^2}\right)+\frac{\bar{t}}{(l_1+l_2)(l_2+t)}\left[2\nu\left(1+\frac{l_1+l_2}{\bar{t}}+\frac{2x}{l_2+t}-\frac{l_1(l_1+l_2)}{\bar{t}(l_2+t)}\right)\right.$$

$$\left.\left.-\frac{\nu}{2-\nu}\frac{2z^2}{(l_1+l_2)(l_2+t)}\left(1-\frac{x}{l_1}+\frac{4x}{l_2+t}+\frac{4x}{l_1+l_2}\right)\right]\right\}.$$